Separability problem for multipartite states of rank at most four

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One of the most important problems in quantum information is the separability problem, which asks whether a given quantum state is separable. We investigate multipartite states of rank at most four which are PPT (i.e., all their partial transposes are positive semidefinite). We show that any PPT state of rank two or three is separable and has length at most four. For separable states of rank four, we show that they have length at most six. It is six only for some qubit-qutrit or multiqubit states. It turns out that any PPT entangled state of rank four is necessarily supported on a $3 \otimes 3$ or a $2 \otimes 2 \otimes 2$ subsystem. We obtain a very simple criterion for the separability problem of the PPT states of rank at most four: such a state is entangled if and only if its range contains no product vectors. This criterion can be easily applied since a four-dimensional subspace in the $3 \otimes 3$ or $2 \otimes 2 \otimes 2$ system contains a product vector if and only if its Plücker coordinates satisfy a homogeneous polynomial equation (the Chow form of the corresponding Segre variety). We have computed an explicit determinantal expression for the Chow form in the former case, while such expression was already known in the latter case.

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Contents

I. INTRODUCTION

In quantum physics, the condition of the spatially distributed systems is described by multipartite quantum states. The systems can be uncorrelated, and the corresponding state is *separable*. Otherwise, the state is entangled [50]. Entangle states are the basic ingredients of quantum-information tasks [5]. The so-called *separablity problem*, namely to distinguish multipartite separable states from entangled states, is then a basic task in quantum information. In this paper, we will address this problem by studying multipartite states ρ of rank at most four, and give a complete answer in this case. We recall that if ρ is separable then it must be PPT i.e., *all partial transposes of ρ are positive semidefinite*. The following separability criterion is based on Lemma 11 and Theorems 15, 22 and 28:

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Any multipartite state of rank less than four is separable if and only if it is PPT. Any multipartite state ρ of rank four is separable if and only if (1) ρ is PPT, and (2) the range of ρ contains a product vector.

We will show below that this criterion is easy to apply in practice. By the Peres-Horodecki criterion [31, 32], the condition (2) above is automatically satisfied by qubit-qubit and qubit-qutrit PPT states of rank four. So the two criteria are equivalent for these states. Moreover, we will show that condition (2) is satisfied by all PPT states of rank four, except for some states supported on $3 \otimes 3$ or $2 \otimes 2 \otimes 2$ systems. For these states, the violation of condition (2) becomes an essential tool for the construction of PPT entangled states of rank four.

In general, we consider a quantum system consisting of n parties $A_1, \ldots, A_n$ with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. We denote the dimension of $\mathcal{H}_i$ by d_i and the dimension of $\mathcal{H}$ by $d := d_1 d_2 \cdots d_n$. We denote by Γ_i the partial transposition operator on the system A_i computed in some fixed o.n. basis. Thus, if $\rho = \rho_1 \otimes \cdots \otimes \rho_i \otimes \cdots \otimes \rho_n$ then $\rho^{\Gamma_i} = \rho_1 \otimes \cdots \otimes \rho_i^T \otimes \cdots \otimes \rho_n$, where the exponent T denotes transposition. We denote by Γ the group generated by the pairwise commuting involutory operators Γ_i. The elements of Γ are the products $\Gamma_S = \prod_{i \in S} \Gamma_i$ where S runs through all subsets of $\{1, 2, \ldots, n\}$. We say that a state ρ on $\mathcal{H}$ has PPT if $\rho^{\Gamma_S} \geq 0$ for all subsets S. Evidently if ρ is PPT then so is ρ^{Γ_S} for any S. If a state ρ is not PPT, we shall say that it is NPT. The range and the rank of any linear operator ρ will be denoted by $\mathcal{R}(\rho)$ and $r(\rho)$, respectively. Unless stated otherwise, the states will not be normalized.

A *product vector* is a nonzero vector of the form $|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$, which will be also written as $|\psi_1, \ldots, \psi_n\rangle$. We say that a subspace of $\mathcal{H}$ is *completely entangled* (CES) if it contains no product vectors. For counting purposes we do not distinguish product vectors which are scalar multiples of each other. A (non-normalized) *pure product state* is the tensor product $|\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_n\rangle\langle\psi_n|$, where $|\psi_i\rangle \in \mathcal{H}_i$ are nonzero vectors. A state is *separable* if it is a finite sum of pure product states. (Some authors refer to these states as *fully separable*.) The *length*, $L(\rho)$, of a separable state ρ is the minimal number of pure product states over all such decompositions of ρ [18]. A state is *entangled* if it is not separable. It is immediate from the definition of separable states that every separable state ρ is PPT. In other words, an NPT state must be entangled. So it is sufficient to consider PPT states when we consider the separability problem.

The separability problem has a complete answer for the pure state $|\psi\rangle$, whose density matrix $\rho = |\psi\rangle\langle\psi|$ has rank one. By definition $|\psi\rangle$ is separable if and only if it is a product vector. This is equivalent to the statement that any single-party reduced density operator of ρ has rank one, which can be easily verified. Nevertheless, the separability problem becomes quickly intractable as the dimension and number of systems increase [26]. As far as we know, there is no complete answer for multipartite states of rank two despite of some partial results for tripartite systems [2, 23, 35].

In Lemma 11 and Theorem 15, we show that any PPT state of rank two or three is separable. In Theorem 22, we show that any PPT entangled state (PPTES) of rank four is necessarily supported on a $3 \otimes 3$ or a $2 \otimes 2 \otimes 2$ subsystem. This is based on the preliminary results including Lemmas 18, 19 and 20. We deduce that any n-partite PPT state with $n \geq 4$ and rank four is separable, see Corollary 21. Recall that two-qutrit PPTES of rank four have been recently fully described, see [6, 10, 11, 47], by using the two-qutrit unextendible product bases (UPB). In particular, the range of such a state is a CES. We obtain a similar result for three-qubits. This is based on Propositions 25 and 26, which give various properties of three-qubit PPTES ρ of rank four. For example, Proposition 26 (ii) states that $\mathcal{R}(\rho)$ is a CES, and (v) states that up to a scalar multiple there is only one PPTES with specified range, $\mathcal{R}(\rho)$. It is known that both of these properties hold for two-qutrit PPTES of rank four [10], see below Theorem 3 (i),(iv). As a corollary of these results, any PPT state of rank at most four is entangled if and only if $\mathcal{R}(\rho)$ is a CES, see Theorem 28. This gives a complete answer to the separability problem for multipartite states of rank at most four. We also construct some criteria for deciding the separability of certain states of arbitrary dimensions in Lemmas 14, 16 and 23.

We emphasize that Theorem 28 can be easily applied to any PPT state of rank at most four. By Lemma 11, Theorem 15 and 22, it is sufficient to consider the two-qutrit or three-qubit PPT state of rank four, otherwise such state is separable. It is known that a four-dimensional subspace in the $3 \otimes 3$ or $2 \otimes 2 \otimes 2$ system contains a product vector if and only if its Plücker coordinates satisfy a homogeneous polynomial equation, known as the Chow form of the corresponding Segre variety. We have computed an explicit determinantal expression for the Chow form of the $3 \otimes 3$ system, see Eq. (37), and we also give the known Chow form for the $2 \otimes 2 \otimes 2$ system in Eq. (38). They vanish on the range of the two-qutrit or three-qubit PPT state of rank four if and only if the state is separable. This is stated in Theorem 31. The verification of Eqs. (37) and (38) can be readily performed by an ordinary computer. Note that our method is analytically operational. It is different from the numerical test employing methods of semi-definite programming and optimization [19]. As a generalization, we also compute the Chow form of the Segre variety $\mathcal{P}^{M-1} \times \mathcal{P}^1$ in Proposition 30.

Let us mention some applications of our results. First, it is known that any PPT state is a sum of extreme PPT states, so it is important to characterize these extreme states [11, 13]. By Lemma 11 and Theorem 15, the PPT states of rank two or three are not extreme. By Theorem 22 and Proposition 29, there are only three types of multipartite extreme PPT states of rank at most four: pure product states, two-qutrit PPTES, and three-qubit PPTES. The last assertion coincides with the recent numerical test as reported in [24]. Furthermore, it is known that some three-qubit

PPTES of rank four can be constructed by using three-qubit UPB [17]. So such states may be related to some Bell inequalities with no quantum violation, which can also be constructed by three-qubit UPB [1]. We also show that any PPTES of rank four is strongly extreme, see Proposition 29. The latter are extreme PPTES whose range does not contain the range of any other PPT state [13], see Definition 9.

Second, the length of a separable state ρ represents the minimal physical effort needed to realize ρ by the entanglement of formation [7]. Two separable states of different lengths are not equivalent under stochastic local operations and classical communications (SLOCC) [22]. In Lemma 11 and Theorem 15, besides the results on separability, we also prove that the multipartite separable states of rank two have length two, and those of rank three have length three or four. We further show that separable states of rank four have length at most six, see Lemma 17. The bound six is reached by some known examples of qubit-qutrit separable states, see [12, Table 2], as well as by some three-qubit separable states, see the proof of Lemma 17 (iii). These results on lengths of separable states generalize some of the recent results obtained in [3, 12, 14, 28, 29]. We may conclude that the bigger the rank is, the more different lengths the separable states could have.

Third, we present another interpretation of our result in quantum information. Recall that a pure state is genuinely entangled if it is not a product state for any bipartition of the systems [46]. We say that a mixed multipartite state ρ is genuinely entangled if for any ensemble $\{p_i, |\psi_i\rangle\}$ realizing this state, i.e., such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, at least one state $|\psi_j\rangle$ is genuinely entangled. Genuine entanglement can be detected by many kinds of entanglement witnesses based on Bell inequalities [46] or stabilizer theory [49]. Proposition 26 and Theorem 28 imply that the multipartite PPT state ρ of rank four contains genuine entanglement if and only if it is a bipartite state. In particular, any pure state of the ensemble generating ρ is entangled by Theorem 3 (i). In this sense, genuine PPTES of rank four are rare and not easily prepared quantum resource.

Fourth, if we regard the state ρ in Lemma 14 (i) as a bipartite state of system A_1 and all other systems $A_2 \cdots A_n$ together, then ρ is a "generalized classical" state as defined in [8, Definition 1]. In the same paper, analytical methods using algorithms and a physical criterion have been given for detection of generalized classical multipartite states. In Lemma 14 (i), we give another method to detect generalized classical PPT states. Let us also mention that the generalized classical states allow for the quantum-information tasks with non-disruptive local state identification. This is related to the so-called local broadcasting [44].

The paper is organized as follows. In Sec. II we state the known facts used throughout this paper. In Sec. III we study multipartite PPT states of rank two and three. The main results are presented in Lemma 11 and Theorem 15. In Sec. IV we study multipartite PPT states of rank four. The main result is stated in Lemma 17 and Theorem 22. In Sec. V we study the three-qubit PPT states of rank four. Based on this result we can characterize all PPTES of rank four. The main results are presented in Proposition 26 and Theorem 28. In Sec. VI, we show how one can verify the CES condition for the separability of multipartite states of rank four by using the Chow form. The main result is stated in Theorem 31.

II. PRELIMINARIES

From now on, for a given multipartite state ρ, we denote by $\rho_{i_1,\ldots,i_k}$ the reduced density operator on the systems $A_{i_1}, \ldots, A_{i_k}$. For any PPT state ρ on $\mathcal{H}$ and any pure state $|\Psi\rangle \in \otimes_{i\in S}\mathcal{H}_i$, the state $\langle\Psi|\rho|\Psi\rangle$ is a PPT state on the space $\otimes_{i\in S'}\mathcal{H}_i$, where S' is the complement of S in $\{1, 2, \ldots, n\}$. Another useful property of multipartite PPT states is that they remain PPT when several local systems are combined into one system. This property enables us to consider multipartite states as bipartite states, and hence simplify many proofs. See for instance, the application of Lemma 13 in the proof of Lemma 14.

We say that a multipartite state ρ is a $r_1 \times r_2 \times \cdots \times r_n$ *state* if its local ranks are $r_1, r_2, \ldots, r_n$, i.e., $r(\rho_i) = r_i$ for each i. For any state ρ on $\mathcal{H}$ and any subset $S \subseteq \{1, \ldots, n\}$, we have

$$\left(\rho^{\Gamma_S}\right)_i = \begin{cases} \rho_i^T, & i \in S; \\ \rho_i, & i \notin S. \end{cases} \tag{1}$$

Consequently,

$$r\left((\rho^{\Gamma_S})_i\right) = r(\rho_i), \quad i = 1, \ldots, n, \tag{2}$$

i.e., the rank of any single system reduced state is invariant under all partial transpositions. If ρ is an $r_1 \times r_2 \times \cdots \times r_n$ state, then ρ^{Γ_S} is too. If ρ is a PPTES so is ρ^{Γ_S}, but they may have different ranks.

As the bipartite case occurs often, we shall use the following simplified notation: $M = d_1$, $N = d_2$, $A = A_1$, $B = A_2$. When ρ is a bipartite state, we refer to the ordered pair $(r(\rho), r(\rho_1^\Gamma))$ as the *birank* of ρ. The birank has been used to characterize many bipartite PPT states. For example, two-qubit and qubit-qutrit separable states have been classified

independently in terms of the birank in [12, Table I,II] and [15, 27], respectively. In the proof of Lemma 16 (ii), we will use the fact that the length of a separable bipartite state is not smaller than the maximum of $r(\rho)$ and $r(\rho_1^\Gamma)$.

Let us now recall some basic results from quantum information regarding the separability and PPT properties of bipartite states. Let us start with the basic definition.

Definition 1 *We say that two n-partite states ρ and σ are* equivalent under SLOCC (SLOCC-equivalent *or just* equivalent) *if there exists an invertible local operator (ILO) $A = \bigotimes_{i=1}^n A_i \in \mathrm{GL} := \mathrm{GL}_{d_1}(\mathbf{C}) \times \cdots \times \mathrm{GL}_{d_n}(\mathbf{C})$ such that $\rho = A\sigma A^\dagger$ [22].*

It is easy to see that any ILO transforms PPT, entangled, or separable state into the same kind of states. We shall often use ILOs to simplify the density matrices of states. For example, any M linearly independent vectors in $\mathbf{C}^M$ can be converted into the o. n. basis $\{|i\rangle_A : i = 0, \ldots, M-1\}$ by an ILO.

Let us extend the formal definition of the term "general position" [13, Definition 7] to the multipartite case.

Definition 2 *We say that an m-tuple of vectors $(v_1, \ldots, v_m)$ in a vector space V is in* general position *if, for any subset $I \subseteq \{1, \ldots, m\}$ with $|I| \le \mathrm{Dim}\, V$, the vectors v_i, $i \in I$, are linearly independent. We say that an m-tuple $(|\phi_{k,1}, \ldots, \phi_{k,n}\rangle)_{k=1}^m$ of product vectors in $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is in* general position *if the m-tuple $(|\phi_{k,j}\rangle)_{k=1}^m$ is in general position in $\mathcal{H}_j$ for each $j = 1, \ldots, n$.*

Let us recall from [9, Theorem 22] and [10, Theorems 17,22,24] the main facts about the 3×3 PPT states of rank four. Let $M = N = 3$ and let $\mathcal{U}$ denote the set of unextendible product bases in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. For $\{\psi\} \in \mathcal{U}$ we denote by $\Pi\{\psi\}$ the normalized state $(1/4)P$, where P is the orthogonal projector onto $\{\psi\}^\perp$.

Theorem 3 ($M = N = 3$) *For a 3×3 PPT state ρ of rank four, the following assertions hold.*

(i) ρ is entangled if and only if $\mathcal{R}(\rho)$ is a CES.

(ii) If ρ is separable, then it is either the sum of four pure product states or the sum of a pure product state and a 2×2 separable state of rank three.

(iii) If ρ is entangled, then

(a) ρ is extreme;

(b) $r(\rho_1^\Gamma) = 4$;

(c) $\rho \propto A \otimes B\ \Pi\{\psi\}\ A^\dagger \otimes B^\dagger$ for some $A, B \in \mathrm{GL}_3(\mathbf{C})$ and some $\{\psi\} \in \mathcal{U}$;

(d) $\ker \rho$ contains exactly 6 product vectors, and these vectors are in general position.

(iv) If the normalized states ρ and ρ' are 3×3 PPTES of rank four with the same range, then $\rho = \rho'$.

In Proposition 26, we will show that the three-qubit PPTES of rank four have similar properties as 3×3 PPTES of rank four. These properties turn out to be essential in proving the extremality of three-qubit PPTES of rank four, see Proposition 29.

From [34, Theorem 1] we have

Theorem 4 *The $M \times N$ states of rank less than M or N are distillable, and consequently they are NPT.*

The next result follows from [9, Theorem 10], [33] and Theorem 4, see also [9, Proposition 6 (ii)].

Proposition 5 *Let ρ be an $M \times N$ state of rank N.*

(i) If ρ is PPT, then it is a sum of N pure product states. Consequently, $r(\sigma) > \max(r(\sigma_A), r(\sigma_B))$ for any PPTES σ, any bipartite PPT state of rank ≤ 3 is separable, and the bipartite PPTES of rank four must be supported on $3 \otimes 3$.

(ii) If ρ is NPT, then it is distillable.

By Theorem 4, in case (i) we must have $M \le N$. We will generalize the last-but-one assertion of (i) to multipartite PPT states in Lemma 11 and Theorem 15. The final assertion of (i) will be generalized to three-qubit case in Theorem 22. The proof of these results require the notion of reducible and irreducible states constructed in [9, Definition 11].

Definition 6 *We say that a linear operator $\rho : \mathcal{H} \to \mathcal{H}$ is an* A-direct sum *of linear operators $\alpha : \mathcal{H} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{H}$, and we write $\rho = \alpha \oplus_A \beta$, if $\mathcal{R}(\rho_A) = \mathcal{R}(\alpha_A) \oplus \mathcal{R}(\beta_A)$. A bipartite state ρ is* A-reducible *if it is an A-direct sum of two states; otherwise ρ is* A-irreducible. *One defines similarly the* B-direct sum $\rho = \alpha \oplus_B \beta$, *the* B-reducible *and the* B-irreducible *states. We say that a state ρ is* reducible *if it is either A or B-reducible. We say that ρ is* irreducible *if it is not reducible. We write $\rho = \alpha \oplus \beta$ if $\rho = \alpha \oplus_A \beta$ and $\rho = \alpha \oplus_B \beta$, and in that case we say that ρ is a* direct sum *of α and β.*

The next result on reducible states is from [13, Lemma 15].

Lemma 7 *Let α and β be linear operators on $\mathcal{H}$.*
(i) If $\rho = \alpha \oplus_B \beta$, then $\rho_1^\Gamma = \alpha_1^\Gamma \oplus_B \beta_1^\Gamma$.
(ii) If α and β are Hermitian and $\rho = \alpha \oplus_A \beta$, then $\rho_1^\Gamma = \alpha_1^\Gamma \oplus_A \beta_1^\Gamma$.
(iii) If a PPT state ρ is reducible, then so is ρ_1^Γ.

Let us recall a related result [9, Corollary 16]. It will be used in the proof of Lemma 16 (i).

Lemma 8 *Let $\rho = \sum_i \rho^{(i)}$ be an A or B-direct sum of the states $\rho^{(i)}$. Then ρ is separable [PPT] if and only if each $\rho^{(i)}$ is separable [PPT]. Consequently, ρ is a PPTES if and only if each $\rho^{(i)}$ is PPT and at least one of them is entangled.*

The set of normalized multipartite PPT states is a compact convex set. We refer to its extreme points as *extreme states*. More generally, for a non-normalized PPT state ρ we say that it is *extreme* if the normalization of ρ is an extreme state. The following definition generalizes that in [13, Definition 4].

Definition 9 *The multipartite PPT state σ is* strongly extreme *if there are no PPT states $\rho \neq \sigma$ such that $\mathcal{R}(\rho) = \mathcal{R}(\sigma)$.*

Obviously any pure product state is strongly extreme. By applying the proof of [13, Lemma 19] to multipartite states, we obtain that any strongly extreme state ρ is extreme and that $\mathcal{R}(\rho)$ is a CES if $r(\rho) > 1$. It follows from Theorem 3 (iv) that any 3×3 PPTES of rank four is strongly extreme. In fact, we will show in Proposition 29 that any PPTES of rank four is strongly extreme.

III. MULTIPARTITE PPT STATES OF RANK TWO OR THREE

In this section we begin our investigation of multipartite PPT states of small ranks. For PPT states of rank two or three, we will show that they are separable (see Lemma 11 and Theorem 15). We shall also determine their lengths, generalizing the results obtained in [14] and [12].

Any PPT multipartite pure state is a product state and thus has length one. Usually, multipartite states are assumed to have local ranks larger than one. For the sake of completeness, we consider also the states whose local ranks may be one. The following observation is clear.

Lemma 10 *Suppose ρ is an n-partite state with $r(\rho_1) = 1$ and so $\rho = |a_1\rangle\langle a_1| \otimes \sigma_{A_2 \cdots A_n}$. Then*
(i) $r(\rho) = r(\sigma)$.
(ii) ρ is PPT if and only if σ is PPT.
(iii) If ρ is separable, then so is σ and $L(\rho) = L(\sigma)$.

Consequently, when dealing with the separablity of ρ, its rank, or the PPT property one can assume that all local ranks are bigger than one.

We begin with states of rank two. The following lemma generalizes [35, Lemma 4].

Lemma 11 *Any multipartite PPT state of rank two is separable and has length two.*

Proof. We use induction on n, the number of parties. The assertion is trivial if $n = 1$. Assume $n > 1$ and let ρ be PPT state of rank two. By Theorem 4, we must have $r(\rho_1) \leq 2$. If this rank is one, the assertion follows immediately from the induction hypothesis. Hence, we may assume that $r(\rho_1) = 2$. By Proposition 5 (i) we have $\rho = |a\rangle\langle a|_{A_1} \otimes |\varphi\rangle\langle\varphi|_{A_2 \cdots A_n} + |b\rangle\langle b|_{A_1} \otimes |\psi\rangle\langle\psi|_{A_2 \cdots A_n}$, where $|a\rangle, |b\rangle$ are linearly independent. Since this is an A_1-direct sum, the two summands on the right hand side must be PPT states by Lemma 7 and so $|\varphi\rangle$ and $|\psi\rangle$ are product vectors. $\square$

The second assertion of this lemma has been obtained in [21, Lemma 4 (iii)]. From this lemma we deduce another simple fact about states of rank two. (This fact is also a corollary of [9, Theorem 5].)

Lemma 12 *If $|\varphi\rangle \in \mathcal{H}$ is a product vector and $|\psi\rangle \in \mathcal{H}$ is entangled, then the state $\rho = |\varphi\rangle\langle\varphi| + |\psi\rangle\langle\psi|$ is NPT.*

Proof. Assume that ρ is PPT. By Lemma 11 we have $\rho = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$, where $|\alpha\rangle = |a_1, \ldots, a_n\rangle$ and $|\beta\rangle = |b_1, \ldots, b_n\rangle$ are product vectors. Since $|\psi\rangle \in \mathcal{R}(\rho)$ is not a product vector, there are at least two indexes i such that $|b_i\rangle$ is not a scalar multiple of $|a_i\rangle$. It follows that $\mathcal{R}(\rho)$ contains only two product vectors and so we have, say, $|\varphi\rangle = c|\alpha\rangle$. Thus $|\psi\rangle\langle\psi| = (1 - |c|^2)|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$. Since the right hand side is positive semidefinite, we must have

$|c| \leq 1$. Hence, the right hand side is a separable state and we have a contradiction. We conclude that ρ must be NPT. $\square$

We need the following lemma about the bipartite case.

Lemma 13 *Let $\rho = \sum_{i=0}^{l} |a_i, b_i\rangle\langle a_i, b_i|$ be a bipartite A-irreducible separable state of rank $r_1 + 1$ where $r_1 := r(\rho_1)$. The equality $L(\rho) = r_1 + 1$ holds if (i) $r(\rho_1^\Gamma) = r_1 + 1$ or (ii) $|a_0\rangle \propto |a_1\rangle$ and $|b_0\rangle$ and $|b_1\rangle$ are linearly independent.*

Proof. (i) Let $\sigma(t) = \rho - t|a_0, b_0\rangle\langle a_0, b_0|$ for real t. Since $\rho_1^\Gamma = \sum_{i=0}^{l} |a_i^*, b_i\rangle\langle a_i^*, b_i|$ and $r(\rho) = r(\rho_1^\Gamma) = r_1 + 1$, there is a $t_0 \geq 1$ such that $\sigma(t) \geq 0$ and $\sigma(t)_1^\Gamma \geq 0$ for $t \leq t_0$, and $r(\sigma(t)) = r(\sigma(t)_1^\Gamma) = r_1 + 1$ for $t < t_0$ while at $t = t_0$ we have $\min(r(\sigma(t_0)), r(\sigma(t_0)_1^\Gamma)) = r_1$. By Lemma 7 (iii), ρ_1^Γ is also A-irreducible. Thus, we may assume that $r(\sigma(t_0)) = r_1$. The equality $\rho_1 = \sigma(t_0)_1 + t_0\|b_0\|^2|a_0\rangle\langle a_0|$ and the fact that ρ is A-irreducible imply that $r(\sigma(t_0)_1) = r_1$. By Proposition 5, $\sigma(t_0)$ is separable of length r_1, and so $L(\rho) = r_1 + 1$.

(ii) We may assume that $|a_0\rangle = |a_1\rangle$, and also that $\{|a_i\rangle : 1 \leq i \leq r_1\}$ is a basis of $\mathcal{R}(\rho_1)$. As $|b_0\rangle$ and $|b_1\rangle$ are linearly independent, $\{|a_i, b_i\rangle : 0 \leq i \leq r_1\}$ is a basis of $\mathcal{R}(\rho)$. For $j > r_1$ we have $|a_j\rangle = \sum_{i=1}^{r_1} \eta_i |a_i\rangle$ and $|a_j, b_j\rangle = \sum_{i=0}^{r_1} \xi_i |a_i, b_i\rangle$, where ξ_i, η_i are some scalars. It follows that $\xi_i|b_i\rangle = \eta_i|b_j\rangle$ for $1 < i \leq r_1$ and $\xi_0|b_0\rangle + \xi_1|b_1\rangle = \eta_1|b_j\rangle$. If $\eta_i \neq 0$ for some $i > 1$, then $|b_j\rangle \propto |b_i\rangle$. If $\eta_1 \neq 0$ then $|b_j\rangle = \eta_1^{-1}(\xi_0|b_0\rangle + \xi_1|b_1\rangle)$. Since $|a_j^*, b_j\rangle = \sum_{i=1}^{r_1} \eta_i^* |a_i^*, b_j\rangle$, it follws that $|a_j^*, b_j\rangle$ lies in the span of the vectors $|a_i^*, b_i\rangle$, $i = 0, \ldots, r_1$. Since this holds for all $j > r_1$ and the vectors $|a_i^*, b_i\rangle$, $0 \leq i \leq r_1$, are linearly independent and belong to $\mathcal{R}(\rho_1^\Gamma)$, we conclude that $r(\rho_1^\Gamma) = r_1 + 1$. Hence, (ii) reduces to (i) and the proof is completed. $\square$

We often regard a multipartite state as a bipartite state by considering the parties $A_2, \ldots, A_n$ as a single party. The following lemma is useful in the analysis of such states.

Lemma 14 *Let $\rho = \sum_{i=0}^{l} |a_i\rangle\langle a_i|_{A_1} \otimes (\sigma^{(i)})_{A_2 \cdots A_n}$ be a PPT state, $n \geq 3$ and $r_k := r(\rho_k) > 1$ for each k.*

(i) If $l = r_1 - 1$ and $r(\sigma^{(i)}) \leq 2$ for each i, then each $\sigma^{(i)}$ is separable and $L(\rho) = r(\rho)$.

(ii) If $r(\rho) = r_1 + 1$, then ρ is separable. If we further assume that ρ is A_1-irreducible, then $L(\rho) = r_1 + 1$.

Proof. (i) Since $l = r_1 - 1$, the $|a_i\rangle$ are linearly independent. Let $j \in \{0, \ldots, r_1 - 1\}$ be arbitrary and choose $|b_j\rangle \in \mathcal{H}_1$ such that $\langle a_i|b_j\rangle = \delta_{ij}$ for all i. Since $\langle b_j|\rho|b_j\rangle = \sigma^{(j)}$, we conclude that $\sigma^{(j)}$ is PPT. By Lemma 11, $\sigma^{(j)}$ is separable and $L(\sigma^{(j)}) = r(\sigma^{(j)})$. Since j is arbitrary, ρ is separable. Since $r(\rho) = \sum_{i=0}^{l} r(\sigma^{(i)}) = \sum_{i=0}^{l} L(\sigma^{(i)}) \geq L(\rho) \geq r(\rho)$, we have $L(\rho) = r(\rho)$.

(ii) We shall use induction on r_1. If $r_1 = 1$, the assertion is true by Lemma 11. Now let $r_1 > 1$. Assume that ρ is A_1-reducible. Thus, we have $\rho = \theta \oplus_{A_1} \chi$. Since ρ is biseparable for the partition $A_1 : A_2 \cdots A_n$, the same is true for θ and χ. We have $r_1 + 1 = r(\theta) + r(\chi)$ and $r_1 = r(\theta_1) + r(\chi_1)$. Since ρ is PPT, the states θ and χ are also PPT. Theorem 4 implies that, say, $r(\theta) = r(\theta_1)$ and $r(\chi) = r(\chi_1) + 1$. Then part (i) and Proposition 5 imply that θ is a sum of $r(\theta_1)$ pure bipartite product states. By part (i), θ is separable. So it remains to show χ is separable. This is true when $r(\chi_1) = 1$ by Lemma 11. So $r(\chi_1) > 1$, and the fact $r(\chi) = r(\chi_1) + 1$ implies that there is at least one $i > 1$ such that $r_i > 1$. If there is really only one such i, then χ is separable. On the other hand if there are two different $i, j > 1$ such that $r_i, r_j > 1$, then χ is separable by induction hypothesis. So the assertion holds.

From now on we assume that ρ is A_1-irreducible. Without any loss of generality, we may assume that $\sigma^{(i)} = |\psi_i\rangle\langle\psi_i|$ for each i. Note that we must have $l \geq r_1$. We may also assume that the $|a_i\rangle$ with $i < r_1$ form a basis of $\mathcal{R}(\rho_1)$ and, by applying an ILO, we may assume that $|a_i\rangle = |i\rangle$ for $i < r_1$. We may assume that the representation $\rho = \sum_{i=0}^{l} |a_i, \psi_i\rangle\langle a_i, \psi_i|$ is chosen so that l is minimal. In particular, no two $|a_i, \psi_i\rangle$ are parallel.

Suppose that ρ is entangled, and we will derive a contradiction. Some $|\psi_i\rangle$ must be entangled, say $|\psi_0\rangle$. For convenience, we can assume that $r(\sigma_2^{(0)}) > 1$. Since

$$r(\rho_{12}) \geq \sum_{i=0}^{r_1 - 1} r(\sigma_2^{(i)}) \geq 2 + (r_1 - 1) = r_1 + 1, \tag{3}$$

we have $r(\rho_{12}) = r_1 + 1$ by Theorem 4. It follows that $r(\sigma_2^{(i)}) = 1$ for $0 < i < r_1$ and $r(\sigma_2^{(0)}) = 2$. Moreover, if $j \geq r_1$ and $\langle i|a_j\rangle \neq 0$ for some $0 < i < r_1$ then we can replace $\sigma^{(i)}$ by $\sigma^{(j)}$ in Eq. (3). Thus, we have $r(\sigma_2^{(i)}) = 1$ for all i for which $|a_i\rangle$ is not parallel to $|0\rangle$.

We claim that $l = r_1$. If two of the $|a_i\rangle$ are parallel, the claim follows from Lemma 13 (ii). We may now assume that the $|a_i\rangle$ are pairwise non-parallel. Consequently, $r(\sigma_2^{(i)}) = 1$ for all $i > 0$. Let us write $\rho = |0, \psi_0\rangle\langle 0, \psi_0| + \rho'$ and let $|\psi_i\rangle = |b_i, \varphi_i\rangle$ for $i > 0$. Since $r(\rho_{12}) = r_1 + 1$ and $|0, \psi_0\rangle$ is entangled w.r.t. the bipartition $A_1A_2 : A_3 \cdots A_n$, [9, Theorem 5] implies that $r(\rho'_{12}) = r_1 + 1$. Consequently, we may assume that $\mathcal{R}(\rho)$ is spanned by the $|a_i, b_i, \varphi_i\rangle$ with $1 \leq i \leq r_1 + 1$. We can now write $|0, \psi_0\rangle = \sum_{i=1}^{r_1+1} x_i |a_i, b_i, \varphi_i\rangle$, as well as $|a_{r_1}\rangle = \sum_i c_i |i\rangle$ and $|a_{r_1+1}\rangle = \sum_i d_i |i\rangle$.

Since $|a_i\rangle = |i\rangle$ for $i < r_1$ and $|a_{r_1}\rangle$ is not parallel to $|0\rangle$, we may assume that for some $1 \le k < r_1$ we have $c_i \ne 0$ for $1 \le i \le k$ and $c_i = 0$ for $k < i < r_1$. So for $j = 1, \ldots, k$, we have

$$\begin{aligned} |\psi_0\rangle &= x_{r_1} c_0 |b_{r_1}, \varphi_{r_1}\rangle + x_{r_1+1} d_0 |b_{r_1+1}, \varphi_{r_1+1}\rangle, && (4) \\ -x_j |b_j, \varphi_j\rangle &= x_{r_1} c_j |b_{r_1}, \varphi_{r_1}\rangle + x_{r_1+1} d_j |b_{r_1+1}, \varphi_{r_1+1}\rangle. && (5) \end{aligned}$$

Since $|\psi_0\rangle$ is entangled w.r.t. the partition $A_2 : A_3 \cdots A_n$, we have $x_{r_1} x_{r_1+1} c_0 d_0 \ne 0$. As $c_j \ne 0$, we must have $x_j \ne 0$, $d_j = 0$ and $|b_j, \varphi_j\rangle \propto |b_{r_1}, \varphi_{r_1}\rangle$. Thus

$$\begin{aligned} \rho &= |0, \psi_0\rangle\langle 0, \psi_0| + \sum_{j=1}^{k} |a_j, b_j, \varphi_j\rangle\langle a_j, b_j, \varphi_j| + |a_{r_1}, b_{r_1}, \varphi_{r_1}\rangle\langle a_{r_1}, b_{r_1}, \varphi_{r_1}| + \rho'' \\ &= |0, \psi_0\rangle\langle 0, \psi_0| + (y|0\rangle\langle 0| + \chi)|b_{r_1}, \varphi_{r_1}\rangle\langle b_{r_1}, \varphi_{r_1}| + \rho'', && (6) \end{aligned}$$

with $y \ne 0$, χ positive semidefinite, and ρ'' biseparable for the partition $A_1 : A_2 \cdots A_n$. Now Lemma 13 (ii) shows that l must be equal to r_1. Thus, our claim is proved.

Since $l = r_1$ and ρ is A_1-irreducible, we obtain $r(\sigma_2^{(i)}) = 1$ for $i > 0$. Lemma 12 implies that the state $\langle 0|_{A_1} \rho |0\rangle_{A_1}$ is NPT with respect to the partition $A_2 : A_3 \cdots A_m$. Hence, we have a contradiction. Thus, ρ must be separable.

Finally we prove the second assertion of (ii). Let $|\psi_i\rangle = |a_{i,2}, \ldots, a_{i,n}\rangle$ for all i. By Theorem 4 and Proposition 5, $r(\rho_{12}) = r_1$ or $r_1 + 1$. In the former case, $\mathcal{R}(\rho_{12})$ is spanned by the $|a_i, a_{i,2}\rangle$, $i < r_1$. Since any $|a_i, a_{i,2}\rangle \in \mathcal{R}(\rho_{12})$ and $r_2 > 1$, we see that ρ is A_1-reducible and we have a contradiction. Thus $r(\rho_{12}) = r_1 + 1$, and we may assume that $\mathcal{R}(\rho)$ is spanned by the $|a_i, \psi_i\rangle$, $i \le r_1$. Using an ILO we may assume $|a_{r_1}\rangle = \sum_{i=0}^{s-1} |i\rangle$, $s \le r_1$. We divide the set of integers $0, 1, \cdots, r_1$ into $k \ge 2$ disjoint subsets $S_1, \cdots, S_k$. Any two vectors $|a_{i,3}, \ldots, a_{i,n}\rangle$ and $|a_{j,3}, \ldots, a_{j,n}\rangle$ are linearly independent if and only if i, j are from different subsets S_i, S_j. Since any $|a_i, \psi_i\rangle \in \mathcal{R}(\rho)$, Proposition 5 implies that for $j > r_1$, we have $|a_{j,3}, \ldots, a_{j,n}\rangle \propto |a_{i,3}, \ldots, a_{i,n}\rangle$ for some $i \le r_1$. The state can be written as

$$\rho = \sum_{i=1}^{k} \Big(\rho^{(i)} \otimes |a_{i,3}, \cdots, a_{i,n}\rangle\langle a_{i,3}, \cdots, a_{i,n}| \Big), \tag{7}$$

where $\rho^{(i)}$ are bipartite separable states and $\mathcal{R}(\rho^{(i)})$ is spanned by $|a_i, a_{i2}\rangle, i \in S_i$. So $\sum_i r(\rho^{(i)}) = r(\rho_{12}) = r(\rho)$.

Since ρ is A_1-irreducible and $k \ge 2$, no set S_i consists of $0, \cdots, s-1, r_1$. So $r(\rho^{(i)}) = r(\rho_1^{(i)})$ for any i. We have $L(\rho^{(i)}) = r(\rho^{(i)})$ by Proposition 5. Therefore $r(\rho) \le L(\rho) \le \sum_i L(\rho^{(i)}) = r(\rho)$. So the proof for the second assertion of (ii) is completed. □

We shall see later (see Lemma 16) that the inequalities $r(\sigma^{(i)}) \le 2$ in part (i) of the above lemma can be replaced with $r(\sigma^{(i)}) \le 3$. We also point out that the length of a reducible state may be bigger than its rank. An example is the tripartite separable state $\rho = (|000\rangle\langle 000| + |111\rangle\langle 111|) \oplus_{A_1} (\sigma \otimes |0\rangle\langle 0|)$, where σ is a two-qubit separable state of birank $(3, 4)$ (see [12, Table 1]). So $r(\rho) = r_1 + 1 = 5 < L(\rho) = r(\rho_1^\Gamma) = 6$. Hence the second assertion of (ii) fails for reducible states.

Next, we consider states of rank three. We shall prove that, for any number of parties, any PPT state of rank three is separable.

Theorem 15 *Let ρ be an n-partite PPT state of rank three, $n > 1$, and let $r_k := r(\rho_k) > 1$ for each k. Then*
(i) ρ is separable;
(ii) $L(\rho) \in \{3, 4\}$ and if $L(\rho) = 4$ then $n = 2$ and $r_1 = r_2 = 2$.

Proof. By Theorem 4, we have $r_i \le 3$ and so $r_i \in \{2, 3\}$ for each i.

(i) Proposition 5 (i) implies that $\rho = \sum_i |a_i\rangle\langle a_i|_{A_1} \otimes (\sigma^{(i)})_{A_2 \cdots A_n}$. Thus (i) holds if $n = 2$, so let $n > 2$. If $r_1 = 3$, (i) follows from Proposition 5 (i) and Lemma 14 (i). If $r_1 = 2$, (i) follows from Lemma 14 (ii). Hence, (i) has been proved.

(ii) Recall that the length of a separable state is always greater than or equal to its rank. Thus we have $L(\rho) \ge 3$. Suppose ρ is reducible, say $\rho = \theta \oplus_{A_1} \chi$. Since $r(\theta), r(\chi) \le 2$, we have $3 \le L(\rho) \le L(\theta) + L(\chi) = 3$ by Lemma 11. If $r_1 = 3$ then $L(\rho) = 3$ by Proposition 5 (i). So we can assume ρ is irreducible and $r_i = 2$ for all i. If $n \ge 3$ then $L(\rho) = 3$ by Lemma 14 (ii). Finally, if $n = 2$ then (ii) follows from [12, Table 1]. □

Part (i) may be restated as follows: any n-partite PPTES must have rank at least four.

While bipartite separable states of rank two always have length two (see Lemma 11), the ones of rank three may have length four (see [12, Table 1]).

IV. MULTIPARTITE PPT STATES OF RANK FOUR

We now begin the study of states of rank four. The main result is Theorem 22, where we show that there exist only two types of multipartite PPTES of rank four. They are either $2 \times 2 \times 2$ or 3×3 states.

Let us begin with the reducible PPT states.

Lemma 16 *(i) Any reducible* $2 \times 2 \times 2$ *or* $2 \times 2 \times 3$ *PPT state is separable.*

(ii) Any reducible n-partite PPT state of rank four is separable of length at most five. The bound five is sharp.

(iii) The first assertion of Lemma 14 (i) remains valid when we replace "$r(\sigma^{(i)}) \le 2$*" with "*$r(\sigma^{(i)}) \le 3$*".*

Proof. (i) Suppose ρ is a reducible $2 \times 2 \times 2$ or $2 \times 2 \times 3$ PPT state, say $\rho = \alpha \oplus_{A_1} \beta$. Since ρ is PPT, so are α and β (see Lemma 8). Hence, they are separable by the Peres-Horodecki criterion. The case where ρ is a $2 \times 2 \times 3$ PPT state and $\rho = \alpha \oplus_{A_3} \beta$ can be treated similarly.

(ii) Suppose ρ is a reducible PPT state of rank four, say $\rho = \alpha \oplus_{A_1} \beta$. Then both α and β are PPT and $r(\alpha)+r(\beta) = 4$. By Lemma 11 and Theorem 15, α and β are separable and $L(\rho) \le L(\alpha) + L(\beta) \le \max(2+2, 4+1) = 5$. The bound five is reached by the tripartite separable state $\rho = |000\rangle\langle 000| \oplus_{A_1} (\sigma \otimes |1\rangle\langle 1|)$, where σ is a two-qubit separable state of birank $(3, 4)$ (see [12, Table 1]).

(iii) This follows from the proof of Lemma 14 (i) by using Theorem 15 (i). □

Let us point out that reducible $2 \times 3 \times 3$ PPT states as well as reducible bipartite PPT states of rank five may be entangled. As examples, we can take respectively the states $(|0\rangle\langle 0| \otimes \sigma) \oplus_{A_1} |100\rangle\langle 100|$ and $\sigma \oplus_{A_1} |30\rangle\langle 30|$, where σ is a bipartite 3×3 PPTES of rank four.

In analogy with Lemma 14, we can consider tripartite PPT states $\rho = \sum_i |\psi_i\rangle\langle\psi_i|_{A_1A_2} \otimes |c_i\rangle\langle c_i|_{A_3}$ with $r(\rho) = r(\rho_{1,2}) + 1$. However such states may be entangled. As an example we can take the state $\rho = \sigma \otimes |0\rangle\langle 0| + \epsilon|00\rangle\langle 00| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$, where σ is a 3×3 PPTES of rank four and $\epsilon > 0$ is small. Indeed, such ρ is a $3 \times 3 \times 2$ PPTES with $r(\rho) = 6$ and $r(\rho_{1,2}) = 5$.

We can now obtain sharp upper bounds for the lengths of separable states of rank four.

Lemma 17 *Let* ρ *be an n-partite separable state of rank four. For the sake of simplicity assume that* $n > 1$ *and that the ranks* $r_i := r(\rho_i)$ *satisfy the inequalities* $1 < r_1 \le r_2 \le \cdots \le r_n$*.*

(i) If $n = 2$*, then* $L(\rho) \le 6$ *and equality holds only if* ρ *is a* 2×3 *state.*

(ii) If $n > 2$ *and* $r_n > 2$*, then* $L(\rho) \le 5$ *and equality holds only if* ρ *is reducible.*

(iii) If $n > 2$ *and* $r_n = 2$*, then* $L(\rho) \le 6$ *and equality holds only if* ρ *is the sum of pure product states whose generating product vectors are in general position.*

Proof. Since ρ is separable, we have $r_n \le 4$. If $r_n = 4$ then $L(\rho) = 4$ by Proposition 5 (i) and Lemma 14 (i). Thus, we may assume that $r_n \le 3$.

(i) If $r_1 = 3$ then $L(\rho) \le 5$ by [12, Lemma 16]. If $r_2 = 2$ then $L(\rho) \le 4$ by [12, Table I]. If $r_1 = 2$ and $r_2 = 3$ then $L(\rho) \le 6$ by [12, Table II], where an example with $L(\rho) = 6$ was constructed.

(ii) As we assumed that $r_n \le 3$, we must have $r_n = 3$. For irreducible ρ, after switching the parties A_1 and A_n, Lemma 14 (ii) implies that $L(\rho) = 4$. On the other hand for reducible ρ, the assertion follows from Lemma 16 (ii).

(iii) We may assume now that each $\mathcal{H}_i$ has dimension two. If ρ is reducible then $L(\rho) \le 5$ by Lemma 16 (ii). So, we assume that ρ is irreducible. Let m be an integer such that $1 < m < n$. For $1 \le i_1 < \cdots < i_m \le n$ we shall denote by $r_{i_1,\ldots,i_m}$ the rank of the reduced state $\rho_{i_1,\ldots,i_m}$. We can write $\rho = \sum_{i=1}^{l} |\psi_i\rangle\langle\psi_i|$, where $l \ge 4$ and the $|\psi_i\rangle = |a_{i,1}, \ldots, a_{i,n}\rangle$ are product vectors. Note that $r_{i_1,\ldots,i_m} \in \{2, 3, 4\}$ and so we shall consider three cases.

Case 1: $r_{1,\ldots,m} = 2$ for some choice of indexes $i_1 < \cdots < i_m$. We may assume that $i_k = k$ for $k = 1, \ldots, m$. Then $\mathcal{R}(\rho_{1,\ldots,m})$ is spanned by two of the $|\psi_i\rangle$, say those for $i = 1, 2$. Since each $r_i = 2$, the vectors $|a_{1,j}\rangle$ and $|a_{2,j}\rangle$ must be linearly independent for each $j \in \{1, \ldots, m\}$. Consequently, there are no other product vectors in $\mathcal{R}(\rho_{1,\ldots,m})$. It follows that ρ is A_1-reducible and we have a contradiction.

Case 2: $r_{1,\ldots,m} = 4$ for some choice of indexes $i_1 < \cdots < i_m$. We may assume that $i_k = k$ for $k = 1, \ldots, m$ and that $m = n - 1$. Let $|\psi_i'\rangle = |a_{i,1}, \ldots, a_{i,n-1}\rangle$ and note that $\mathcal{R}(\rho_{1,\ldots,m})$ is spanned by four $|\psi_i'\rangle$, say those for $i = 1, 2, 3, 4$. It follows that the $|\psi_i\rangle$, $i = 1, 2, 3, 4$, span $\mathcal{R}(\rho)$. Consequently, at least two of the $|a_{i,n}\rangle$, $i = 1, 2, 3, 4$, are linearly independent. Let k be the maximum number of pairwise non-parallel vectors among $|a_{i,n}\rangle$, $i = 1, 2, 3, 4$. As $r_n = 2$ we have $k > 1$. Since ρ is irreducible, we must have $k > 2$. It follows that $\rho = \sum_{s=1}^{k} \rho^{(s)} \otimes |a_{s,n}\rangle\langle a_{s,n}|$, where each $\rho^{(s)}$ is an $(n-1)$-partite separable state and $r(\rho^{(1)}) + \cdots + r(\rho^{(k)}) = 4$. As $k > 2$, we have $r(\rho^{(s)}) \le 2$ for all s. By Lemma 11, we have $L(\rho) \le \sum_{s=1}^{k} L(\rho^{(s)}) \le 4$.

Case 3: $r_{i_1,\ldots,i_m} = 3$ for any sequence $i_1 < \cdots < i_m$. We may assume that the $|\psi_i\rangle$, $i = 1, 2, 3, 4$, span $\mathcal{R}(\rho)$. For each $j \in \{1, \ldots, n\}$ denote by k_j the maximum size of a subset $X_j \subseteq \{1, 2, 3, 4\}$ such that all $|a_{i,j}\rangle$ with $i \in X_j$ are

parallel to each other. Let $k = \max(k_1, \dots, k_n)$ and note that $k < 4$. Without any loss of generality we may assume that $k_1 = k$ and that $|a_{1,1}\rangle = \cdots = |a_{k,1}\rangle$. For any i let $|\psi_i''\rangle = |a_{i,2}, \dots, a_{i,n}\rangle$. We have three subcases.

First, let $k = 3$. Since $r_{2,3,\dots,n} = 3$, the vectors $|\psi_i''\rangle$, $i = 1, 2, 3$, span $\mathcal{R}(\rho_{2,3,\dots,n})$. If $|a_{i,1}\rangle$ and $|a_{1,1}\rangle$ span $\mathcal{H}_1$, then $|\psi_i''\rangle \propto |\psi_4''\rangle$. Thus, we have $\rho = |a_1\rangle\langle a_1| \otimes \rho'' + \rho' \otimes |\psi_4''\rangle\langle\psi_4''|$ for some $(n-1)$-partite separable state ρ'' of rank three and some $\rho' \geq 0$. Since $\rho' = p|a_1\rangle\langle a_1| + q|a'\rangle\langle a'|$ where $p \geq 0$, $q > 0$ and $|a'\rangle \neq 0$, ρ is A_1-reducible, and so we have a contradiction.

Second, let $k = 2$. We may assume that

$$
\begin{aligned}
|\psi_1\rangle &= |0, \dots, 0, 0, \dots, 0\rangle, && (8)\\
|\psi_2\rangle &= |0, \dots, 0, 1, \dots, 1\rangle, && (9)\\
|\psi_3\rangle &= |1, \dots, 1, b_{s+1}, \dots, b_n\rangle, && (10)\\
|\psi_4\rangle &= |b_1, \dots, b_s, b'_{s+1}, \dots, b'_n\rangle. && (11)
\end{aligned}
$$

As $k = 2$, $|0\rangle$ and $|b_i\rangle$ must be linearly independent for $i \leq s$. Since $r_{1j} = 3$ for $j > s$, it follows that $|b'_{s+1}, \dots, b'_n\rangle \propto |b_{s+1}, \dots, b_n\rangle$. If $s = 1$ then, for suitably chosen $|a\rangle \in \mathcal{H}_1$, we have

$$
|0, \psi_3''\rangle\langle 0, \psi_3''| + |a, \psi_3''\rangle\langle a, \psi_3''| = |\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4|. \tag{12}
$$

Thus, we can replace $|\psi_3\rangle$ and $|\psi_4\rangle$ with $|0, \psi_3''\rangle$ and $|a, \psi_3''\rangle$ and so we obtain the first subcase ($k = 3$). Similarly, if $s = n - 1$ we can reduce the problem to the first subcase. Now let $1 < s < n - 1$. Since $r_{1,\dots,s} = 3$, at least one of the vectors $|b_j\rangle$ with $j \leq s$ is not parallel to $|1\rangle$. This condition implies $r_{1,\cdots,s+1} = 4$ and it is a contradiction.

Third, let $k = 1$. So the product vectors $|\psi_i\rangle$ are in general position. Note that in all instances covered so far we had $L(\rho) \leq 5$. By applying an ILO and by using the fact that $r_{i_1,\dots,i_m} = 3$ for $1 < m < n$, we may assume that

$$
|\psi_1\rangle = \lambda|0, \dots, 0\rangle, \quad |\psi_2\rangle = \mu|1, \dots, 1\rangle, \quad |\psi_3\rangle = |e, \dots, e\rangle, \tag{13}
$$

where $|e\rangle = |0\rangle + |1\rangle$ and $\lambda\mu \neq 0$. (We have here identified all the spaces $\mathcal{H}_j$ by using their bases $\{|0\rangle, |1\rangle\}$.) For $j \neq 1$ we have $r_{1j} = 3$, and so $|a_{i1}, a_{ij}\rangle$ is a linear combination of $|00\rangle$, $|11\rangle$ and $|ee\rangle$. An easy computation shows that $|a_{ij}\rangle \propto |a_{i1}\rangle$ must hold. Consequently, $\mathcal{R}(\rho)$ is spanned by symmetric product vectors. Since $r_{2,\dots,n} = 3$ and $r(\rho) = 4$, Proposition 5 (i) implies that $4 \leq r(\rho^{\Gamma_1}) \leq 6$. Let l be the length of ρ when viewed as a bipartite separable state for the partition $A_1 : A_2 \cdots A_n$. By [12, Proposition 12], we have $l = r(\rho^{\Gamma_1}) \leq 6$. Thus, $\rho = \sum_{i=1}^{l} |v_i\rangle\langle v_i|_{A_1} \otimes |\chi_i\rangle\langle\chi_i|_{A_2\cdots A_n}$. Since each $|v_i, \chi_i\rangle$ is a linear combination of symmetric product vectors, it follows that $|v_i, \chi_i\rangle$ is also a symmetric product vector. Hence, $L(\rho) \leq l \leq 6$. The equality $L(\rho) = 6$ may hold. For example, we have $L(\rho) = r(\rho^{\Gamma_1}) = 6$ for the state

$$
\rho = |000\rangle\langle 000| + |111\rangle\langle 111| + \sum_{j=1}^{4} |e_j, e_j, e_j\rangle\langle e_j, e_j, e_j|, \tag{14}
$$

where $|e_1\rangle = |0\rangle + |1\rangle$, $|e_2\rangle = |0\rangle - |1\rangle$, $|e_3\rangle = |0\rangle + e^{\frac{1}{3}\pi i}|1\rangle$ and $|e_4\rangle = |0\rangle + e^{\frac{2}{3}\pi i}|1\rangle$. This completes the proof. $\square$

Lemma 18 *Any $3 \times 3 \times 3$ PPT state of rank four is separable.*

Proof. Let ρ be such a state and set $r_{ij} = r(\rho_{ij})$. Suppose that $r_{23} = r_{13} = r_{12} = 3$. Then by Proposition 5 (i) we have $\rho_{12} = \sum_{i=1}^{3} |a_i, b_i\rangle\langle a_i, b_i|$ and $\rho_{23} = \sum_{i=1}^{3} |b'_i, c_i\rangle\langle b'_i, c_i|$. As ρ is a $3 \times 3 \times 3$ state, each of the triples $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, $\{|b'_i\rangle\}$, $\{|c_i\rangle\}$ spans a 3-dimensional subspace. Consequently, ρ can be written (see [16, p7]) as

$$
\rho = \sum_i \left(\sum_{j=1}^{3} |a_j, b_j, c'_{i,j}\rangle \right) \left(\sum_{j=1}^{3} \langle a_j, b_j, c'_{i,j}| \right). \tag{15}
$$

Note that $\mathcal{R}(\rho_{23})$ contains only three product vectors, namely the $|b'_i, c_i\rangle$. By tracing out system A_1, we deduce that all $|b_j, c'_{i,j}\rangle \in \mathcal{R}(\rho_{23})$. For any j there is at least one i such that $|c'_{i,j}\rangle \neq 0$. By permuting the $|b'_j, c_j\rangle$, we may assume that $|b_j\rangle = |b'_j\rangle$ for each j, and that each $|b_j, c'_{i,j}\rangle$ is a scalar multiple of $|b_j, c_j\rangle$. Hence, we have $\rho = \sum_{j=1}^{3} p_j |a_j, b_j, c_j\rangle\langle a_j, b_j, c_j|$, $p_j > 0$. As ρ has rank four, we have a contradiction. Thus some $r_{ij} = 4$, say $r_{23} = 4$. Then it follows from Proposition 5 (i) that ρ satisfies the conditions of Lemma 14. Hence, ρ is separable by part (ii) of that lemma. $\square$

Lemma 19 *Any $2 \times 3 \times 2$ or $2 \times 3 \times 3$ PPT state of rank four is separable.*

Proof. Let ρ be a $2 \times 3 \times s$ PPT state of rank four with $s \in \{2,3\}$ and set $r_{ij} = r(\rho_{ij})$. By Theorem 4 we have $r_{12}, r_{23} \in \{3,4\}$ and $r_{13} \in \{2,3,4\}$. Let us view ρ as a bipartite state for the partition $A_1 : A_2A_3$. Then the last assertion of Proposition 5 (i) implies that

$$\rho = \sum_{i=1}^{m} |a_i\rangle\langle a_i|_{A_1} \otimes \rho^{(i)}_{A_2A_3}, \tag{16}$$

where the $|a_i\rangle$ are pairwise linearly independent. By Lemma 16 (ii), ρ is separable if $m = 2$. Thus, we assume that $m > 2$.

Let us first consider the case $r_{12} = 3$. Assume that some $\rho^{(i)}$, say $\rho^{(1)}$, is entangled. Then $\mathcal{R}(\rho_{12})$ has a basis $\{|a_1, x\rangle, |a_1, y\rangle, |a_2, z\rangle\}$. Consequently, for any product vector $|u, v\rangle \in \mathcal{R}(\rho_{12})$, $|u\rangle$ must be a scalar multiple of $|a_1\rangle$ or $|a_2\rangle$. It follows from Eq. (16) that $|a_3, w\rangle \in \mathcal{R}(\rho_{12})$ for some $|w\rangle \in \mathcal{H}_2$, and so we have a contradiction. We conclude that all $\rho^{(i)}$ must be separable and so ρ is separable.

From now on we consider the remaining case $r_{12} = 4$. If $s = 3$ then, by Proposition 5 (i), ρ is biseparable for the partition $A_1A_2 : A_3$ and it is separable by Lemma 14. Thus, we are done with the case $s = 3$. We continue with the case $s = 2$. In view of Lemma 16 (ii), we may assume that ρ is irreducible. If $r_{13} = 4$ then Proposition 5 (i), and Lemma 14 imply that ρ is separable. Thus, we assume from now on that $r_{13} < 4$. By Proposition 5 (i) we have

$$\rho = \sum_{i=0}^{3} |\psi_i\rangle\langle\psi_i|_{A_1A_2} \otimes (|c_i\rangle\langle c_i|)_{A_3}. \tag{17}$$

Assume that some $|\psi_i\rangle$, say $|\psi_0\rangle$, is entangled. Then $\mathcal{R}(\rho_{23})$ contains three linearly independent vectors $|x, c_0\rangle$, $|y, c_0\rangle$ and $|b, c_1\rangle$, where we assume (as we may) that $|c_1\rangle$ is not parallel to $|c_0\rangle$. If $r_{23} = 3$, then any product vector in $\mathcal{R}(\rho_{23})$ is parallel to $|b, c_1\rangle$ or equal to $|z, c_0\rangle$ for some $|z\rangle \in \mathcal{H}_2$. This contradicts the irreducibility of ρ. Hence, we must have $r_{23} = 4$. If $|c_0\rangle$ is not parallel to any $|c_i\rangle$, $i > 0$, then $r_{13} < 4$ implies that for $i > 0$ we have $|\psi_i\rangle = |a, b_i\rangle$. As r_{12} is PPT, [9, Theorem 5] gives a contradiction. Consequently, we may assume that $|c_3\rangle = |c_0\rangle$. Since ρ is irreducible, the $|c_i\rangle$ with $i < 3$ are pairwise linearly independent. As $r_{13} < 4$, we must have $|\psi_i\rangle = |a, b_i\rangle$ for $i = 1, 2$. Let $\sigma = |\psi_0\rangle\langle\psi_0| + |\psi_3\rangle\langle\psi_3|$. As $|\psi_0\rangle$ is entangled, we have $r(\sigma_2) \in \{2,3\}$. If $r(\sigma_2) = 3$ then $r_{23} = 4$ implies that $|b_1\rangle \propto |b_2\rangle$ contradicting the fact that ρ is irreducible. Hence, we must have $r(\sigma_2) = 2$. Then we may assume that $|\psi_3\rangle$ is a product vector, and we introduce the separable state $\rho' = \rho - |\psi_0, c_0\rangle\langle\psi_0, c_0|$. Since $r_{23} = 4$ and $r(\rho'_{23}) = 3$, [9, Theorem 5] gives a contradiction.

Hence, we have shown that all $|\psi_i\rangle$ must be product vectors and so ρ is separable. □

Lemma 20 *Suppose ρ is an n-partite PPT state of rank four, $n > 2$, all ranks $r_i := r(\rho_i) > 1$, and $\sum_{i=1}^n r_i > 6$. Then ρ is separable.*

Proof. Theorem 4 implies that $r_i \le 4$. If some $r_i = 4$, then ρ is separable by Proposition 5 (i) and Lemma 14 (i). If some $r_i = 3$, we can combine the system $A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n$ into a single party B. Then Proposition 5 (i) implies that $4 \ge r(\rho_B) \ge 2$. We claim that the tripartite state $\rho_{A_1A_iB}$ is separable. The case $r(\rho_B) = 4$ has been discussed before. For the case $3 \ge r(\rho_B) \ge 2$, we use the fact $2 \le r_1 \le 3$ and apply Lemma 18 and 19. So the claim holds. By replacing A_i by A_1 in Lemma 14 (ii), ρ is separable. So it is sufficient to consider $r_i = 2$ for all i. Then the condition $\sum_{i=1}^n r_i > 6$ implies that $n > 3$. Furthermore we may assume ρ is irreducible by Lemma 16. For $1 \le i_1 < \cdots < i_m \le n$ we shall denote by $r_{i_1,\dots,i_m}$ the rank of the reduced state $\rho_{i_1,\dots,i_m}$. We have $r_{i_1,\dots,i_m} \le 4$ by Theorem 4. Since ρ is PPT, we deduce that all $r_{ij} \in \{2,3,4\}$.

We show that the cases that some $r_{ij} = 2$ or 4 can be reduced to the case that some $r_{ij} = 3$. Suppose some $r_{ij} = 4$, say $r_{12} = 4$. By Proposition 5 (i) we have $\rho = \sum_{i=1}^4 |\alpha_i\rangle\langle\alpha_i|_{A_1A_2} \otimes |\beta_i\rangle\langle\beta_i|_{A_3\cdots A_n}$. By combining A_1 and A_2 into a single party and by applying Lemma 14 (i), we conclude that each $|\beta_i\rangle$ is a product vector. Since ρ is irreducible, we may assume $r_{34} = 3$ or 4. If $r_{34} = 4$, then the assertion will follow from $r_{2,\cdots,n} \le 4$. So it is sufficient to consider $r_{34} = 3$. Next, suppose some $r_{ij} = 2$, say $r_{12} = 2$. Proposition 5 (i) implies that $\rho = \sum_i |\psi'_i\rangle\langle\psi'_i|_{A_1A_2} \otimes \rho^{(i)}_{A_3:\cdots:A_n}$. Since ρ is an irreducible multiqubit PPT state, there is at least one entangled state $|\psi'_i\rangle$. So $r_{23} = 3$ or 4. By these arguments we have proved that it is sufficient to consider $r_{ij} = 3$, say $r_{23} = 3$.

Let us view ρ as a $(n-1)$-partite PPT state σ for the partition $A_2A_3 : A_1 : A_4 : \cdots : A_n$. Since $n > 3$, we may replace r_{23} by r_i in the first paragraph of the proof. So σ is separable, we have $\rho = \sigma = \sum_{i=1}^k |\psi_i\rangle\langle\psi_i|_{A_2A_3} \otimes |\varphi_i\rangle\langle\varphi_i|_{A_1A_4\cdots A_n}$, where $|\varphi_i\rangle$ are product vectors.

We claim that $k = 4$. Since ρ is irreducible, the state σ is reducible if and only if it is A_2A_3-reducible. We have $\sigma = \alpha \oplus_{A_2A_3} \beta$ where α, β are $(n-1)$-partite separable states w.r.t. to the partition $A_2A_3 : A_1 : A_4 : \cdots : A_n$. There are only two cases, i.e., $(r(\alpha), r(\beta)) = (3,1)$ or $(2,2)$. In the former case, the fact ρ is an irreducible multiqubit PPT state

implies that $r(\alpha_{23}) = r(\alpha_i) = 2$, $i = 1, 4, \cdots, n$. Recall that $n \geq 4$, Theorem 15 implies $L(\sigma) \leq L(\alpha)+L(\beta) = 3+1 = 4$. In the latter case, Lemma 11 implies that $L(\sigma) \leq L(\alpha) + L(\beta) = 2 + 2 = 4$. Since $L(\sigma) \geq r(\sigma) = 4$, we have $k = 4$ for the reducible state σ. On the other hand if σ is irreducible, the condition $r_{23} = 3$ and Lemma 14 (ii) imply $k = 4$ too. So the claim has been proved.

Assume some state $|\psi_i\rangle$, say $|\psi_1\rangle$ is entangled. The fact ρ is PPT indicates that $r_{3\cdots n} = 3$ and the states $|\varphi_i\rangle$ are pairwise linearly independent. Suppose $|\varphi_i\rangle$, $i = 1, 2, 3$ form a basis of $\mathcal{R}(\rho_{3\cdots n})$, and we choose $|\varphi\rangle \perp \text{span}\{|\varphi_2\rangle, |\varphi_3\rangle\}$. So the bipartite state $\langle\varphi|\rho|\varphi\rangle = a|\psi_1\rangle\langle\psi_1| + b|\psi_4\rangle\langle\psi_4|$, $a > 0, b \geq 0$ is PPT. Since $|\psi_1\rangle$ is entangled, Lemma 12 implies that $b > 0$ and $|\psi_4\rangle$ is entangled. Since the states $|\varphi_i\rangle$ are pairwise linearly independent, we have $r_{2,\cdots,n} > 4$ and it is a contradiction with Theorem 4. Thus all $|\psi_i\rangle$ must be separable and so is ρ. $\square$

The following is an easy consequence:

Corollary 21 *An n-partite state ρ of rank four, with $n \geq 4$ and all $r(\rho_i) > 1$, is separable if and only if it is PPT.*

Now we can present the main result of this section.

Theorem 22 *Let ρ be an n-partite PPTES of rank four with all $r_i := r(\rho_i) > 1$. Then either $n = 2$ and $r_1 = r_2 = 3$ or $n = 3$ and $r_1 = r_2 = r_3 = 2$. Conversely, in these two cases such PPTES exist.*

Proof. If $n = 2$ then Proposition 5 implies that ρ is a 3×3 state. If $n > 2$ then Lemma 20 implies that ρ is a $2 \times 2 \times 2$ state. For the existence assertion see [17]. $\square$

To conclude this section, we present a lemma and a conjecture beyond the scope of this section.

Lemma 23 *Suppose all $d_1 \times \cdots \times d_n$ PPT states of rank r are separable. Then all $d_1 \times \cdots \times d_n$ PPT states of rank $\leq r$ are separable.*

Proof. Suppose ρ is a $d_1 \times \cdots \times d_n$ PPT state of rank less than r. We can choose a separable state σ such that $\rho + \sigma$ has rank r. Then for any $t > 0$ the state $\rho + t\sigma$ is PPT of rank r, and so it is separable by the hypothesis. As the set of separable states is closed, it follows that ρ is separable. $\square$

Conjecture 24 *Suppose all $d_1 \times \cdots \times d_n$ PPT states of rank r, with each $d_i > 1$ and $n \geq 3$, are separable. Then*
(i) all $(d_1 + 1) \times d_2 \times \cdots \times d_n$ PPT states of rank r are separable;
(ii) all $d_1 \times \cdots \times d_n \times d_{n+1}$ PPT states of rank r are separable.

Note that both assertions are false for $n = 2$. Indeed, all 2×3 and 2×2 PPT states are separable but there exist 3×3 and $2 \times 2 \times 2$ PPTES of rank four.

V. THREE-QUBIT PPTES OF RANK FOUR

In this section we study the properties of three-qubit PPTES of rank four, see Propositions 25 and 26. This result implies that a multipartite PPT state of rank four is entangled if and only if its range contains a product vector, see Theorem 28. In the next section we show that this condition can be easily tested. We also show that any three-qubit PPTES of rank four is strongly extreme in Proposition 29.

Proposition 25 *Suppose ρ is a three-qubit PPTES of rank four. Then $r(\rho_{ij}) = 4$ for all $1 \leq i < j \leq 3$.*

Proof. Suppose that some $r_{ij} := r(\rho_{ij}) < 4$, say $r_{12} < 4$. Then Proposition 5 (i) implies that ρ, viewed as a bipartite state for the partition $A_1 : A_2A_3$, is separable. Thus, we have $\rho = \sum_{i=1}^n |a_i\rangle\langle a_i|_{A_1} \otimes (\rho^{(i)})_{A_2A_3}$, where $n > 1$ and the $|a_i\rangle$ are pairwise nonparallel. If $n = 2$ then ρ is A_1-reducible, and Lemma 16 (i) gives a contradiction. Thus $n > 2$. Since ρ is entangled, we may assume that $\rho^{(1)}$ is entangled. Since $r_{12} < 4$ and $n > 1$, we must have $r_{12} = 3$ and so $\mathcal{R}(\rho_{12})$ is spanned by $|a_1, 0\rangle$, $|a_1, 1\rangle$ and $|a_2, b\rangle$. By applying an ILO we can assume that $|a_1\rangle = |0\rangle$, $|a_2\rangle = |1\rangle$ and $|b\rangle = |0\rangle$. Since the $|a_i\rangle$ are pairwise nonparallel, we have $(\rho^{(i)})_{A_2} \propto |0\rangle\langle 0|$ for $i > 1$ and so

$$\rho = |0\rangle\langle 0|_{A_1} \otimes (\rho^{(1)})_{A_2A_3} + \sum_{i=2}^n |a_i, 0\rangle\langle a_i, 0|_{A_1A_2} \otimes (\rho^{(i)})_{A_3}. \tag{18}$$

Evidently $r(\rho^{(1)}) < 4$, and so we have three cases. We shall prove that each of them leads to a contradiction.

Case 1: $r(\rho^{(1)}) = 1$. Then [9, Theorem 5] implies that ρ_{23} is NPT and we have a contradiction.

Case 2: $r(\rho^{(1)}) = 3$. For $i > 1$, $|0\rangle$ and $|a_i\rangle$ are linearly independent and so $(\rho^{(i)})_{A_3}$ has rank one. By the same argument, any two states $(\rho^{(i)})_{A_3}$ with $i > 1$ are linearly dependent. It follows that ρ is A_1-reducible and Lemma 16 (i) gives a contradiction.

Case 3: $r(\rho^{(1)}) = 2$. Since $\rho^{(1)}$ is a two-qubit entangled state, we have $\rho^{(1)} = |\psi\rangle\langle\psi| + |\beta,\gamma\rangle\langle\beta,\gamma|$. Since ρ_{23} is PPT, [9, Theorem 5] implies that $|0\rangle$ and $|\beta\rangle$ are linearly independent. Let $\sigma = \sum_{i=2}^{n} |a_i\rangle\langle a_i|_{A_1} \otimes (\rho^{(i)})_{A_3}$. As $r(\rho) = 4$, we must have $r(\sigma) > 1$. Hence, $\mathcal{R}(\rho)$ has a basis consisting of vectors $|0,\psi\rangle$, $|0,\beta,\gamma\rangle$, $|1,0,c_1\rangle$ and $|a_i,0,c_2\rangle$ for some $i > 1$. Suppose that $r(\sigma) > 2$. Then, after applying an ILO, we may assume that $\mathcal{R}(\rho)$ contains a vector of the form $(|0\rangle + |1\rangle)|0,x\rangle = y_1|0,\psi\rangle + y_2|0,\beta,\gamma\rangle + y_3|1,0,c_1\rangle + y_4|a_i,0,c_2\rangle$, with $y_1 \neq 0$ or $y_2 \neq 0$. It follows that there is a nonzero vector $|0,x'\rangle \in \mathcal{R}(\rho^{(1)})$. Since $|0\rangle$ and $|\beta\rangle$ are linearly independent, we have a contradiction. We conclude that $r(\sigma) = 2$. Lemma 11 implies that $\sigma = |a_1,c_1\rangle\langle a_1,c_1| + |a_2,c_2\rangle\langle a_2,c_2|$. Next, Lemma 16 (ii) implies that $|a_i\rangle$, $i = 0,1,2$ are pairwise linearly independent. By a similar argument one can show that $|c_1\rangle$ and $|c_2\rangle$ are linearly independent. Thus $r_{13} = 4$ and [9, Theorem 5] implies that the bipartite state $\rho_{2:13}$ is NPT. Hence, we obtain yet another contradiction.

Since we examined all three cases, the proof is completed. □

Given a basis $|\varphi_i\rangle$, $i = 1,\dots,m$, of some Hilbert space, there is a unique basis, say $|\psi_i\rangle$, $i = 1,\dots,m$, such that $\langle\varphi_i|\psi_j\rangle = \delta_{ij}$ for all i,j. We say that two such bases are *reciprocal* to each other.

Proposition 26 *Let ρ be a three-qubit PPTES of rank four. Then*

(i) $r(\rho^{\Gamma_S}) = 4$ for all $S \subseteq \{1,2,3\}$.

(ii) $\mathcal{R}(\rho)$ is a CES.

(iii) We have

$$\rho = \sum_{i=1}^{4} |a_i\rangle\langle a_i|_{A_1} \otimes |\psi_i\rangle\langle\psi_i|_{A_2A_3}, \tag{19}$$

where any two $|a_i\rangle$ are linearly independent, the $|\psi_i\rangle$ are linearly independent and each of them is entangled.

Let $\{|\psi_i'\rangle\}$ be the basis of $\mathcal{H}_2 \otimes \mathcal{H}_3$ reciprocal to $\{|\psi_i\rangle\}$.

(iv) $\mathcal{R}(\rho)$ and $\ker(\rho)$ contain each exactly four bipartite product vectors for the partition $A_1 : A_2A_3$, namely the $|a_i\rangle \otimes |\psi_i\rangle$ and the $(|a_i^\perp\rangle \otimes |\psi_i'\rangle)$, respectively.

(v) Two normalized three-qubit PPTES of rank four having the same range are equal.

Proof. By Lemma 16, ρ is irreducible. By Propositions 5 and 25, we know that ρ can be written as in Eq. (19) with linearly independent $|\psi_i\rangle$.

(i) It is immediate from (19) that the assertion holds if $S = \{1\}$ or $S = \{2,3\}$. Since we can permute the qubits, it holds in general.

(ii) Suppose $\mathcal{R}(\rho)$ is not a CES. Then $\mathcal{R}(\rho)$ contains a bipartite product vector for the partition $A_1 : A_2A_3$. As $\mathcal{R}(\rho)$ is spanned by the vectors $|a_i\rangle \otimes |\psi_i\rangle$ with linearly independent $|\psi_i\rangle$, at least two vectors $|a_i\rangle$ must be parallel, say $|a_1\rangle \propto |a_2\rangle$. Since the 2-dimensional subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$ contains a product vector, we may assume that $|\psi_1\rangle = |\beta,\gamma\rangle$. Thus

$$\rho = |\alpha,\beta,\gamma\rangle\langle\alpha,\beta,\gamma| + \sum_{i=2}^{4} |a_i\rangle\langle a_i|_{A_1} \otimes |\psi_i\rangle\langle\psi_i|_{A_2A_3}, \tag{20}$$

where we have set $|\alpha\rangle = |a_1\rangle$. By applying a similar argument to, say, the partition $A_2 : A_1A_3$, we obtain a formula similar to Eq. (20) in which the first term on the right hand side is replaced by a scalar multiple of $|\beta,\alpha,\gamma\rangle\langle\beta,\alpha,\gamma|$. This observation implies that for any $S \subseteq \{1,2,3\}$ we have $|\alpha,\beta,\gamma\rangle^S \in \mathcal{R}(\rho^{\Gamma_S})$, where $|\alpha,\beta,\gamma\rangle^S$ is the product vector obtained from $|\alpha,\beta,\gamma\rangle$ by replacing its ith factor with the complex conjugate whenever $i \in S$. Consequently, the operator $\sigma := \rho - t|a_1,\alpha,\beta\rangle\langle a_1,\alpha,\beta|$ is a PPT state for small $t > 0$. We can choose $t > 0$ such that for some $S \subseteq \{1,2,3\}$, the state σ^{Γ_S} is still PPT but has rank three. Then Theorem 15 implies that σ^{Γ_S} is separable, and so is ρ. This contradiction proves that $\mathcal{R}(\rho)$ must be a CES.

(iii) Since $|a_i,\psi_i\rangle \in \mathcal{R}(\rho)$, (ii) implies that each $|\psi_i\rangle$ is entangled. If say $|a_1\rangle \propto |a_2\rangle$, then the fact that the 2-dimensional subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$ contains a product vector would contradict (ii).

(iv) This assertion follows immediately from (iii).

(v) By (i) and Proposition 5, we shall use the following analog of Eq. (19):

$$\rho = \sum_{i=1}^{4} |b_i\rangle\langle b_i|_{A_2} \otimes |\varphi_i\rangle\langle\varphi_i|_{A_1A_3}. \tag{21}$$

Let σ be also a PPTES with $\mathcal{R}(\sigma)=\mathcal{R}(\rho)$. Then part (iv) implies that

$$\sigma=\sum_{i=1}^{4} p_i|a_i\rangle\langle a_i|_{A_1}\otimes|\psi_i\rangle\langle\psi_i|_{A_2A_3}=\sum_{i=1}^{4} q_i|b_i\rangle\langle b_i|_{A_2}\otimes|\varphi_i\rangle\langle\varphi_i|_{A_1A_3} \tag{22}$$

where $p_i,q_i>0$. Let A and B be the 8×4 matrices whose columns are the components of the vectors $|a_i\rangle\otimes|\psi_i\rangle$ and $|b_i\rangle\otimes|\varphi_i\rangle$, respectively.

The Eqs. (19) and (21) imply that $AA^\dagger=BB^\dagger$. Similarly, we have $APA^\dagger=BQB^\dagger$, where $P=\mathrm{diag}(p_1,\ldots,p_4)$ and $Q=\mathrm{diag}(q_1,\ldots,q_4)$. We can choose an invertible matrix V such that $VA=[I_4\ 0]^\dagger$. By writing $VB=\begin{bmatrix} X \\ Y\end{bmatrix}$, with X and Y square matrices, the equation $AA^\dagger=BB^\dagger$ implies that $Y=0$ and X is unitary. From the equation $VAPA^\dagger V^\dagger=VBQB^\dagger V^\dagger$ we deduce that $P=XQX^\dagger$. Thus, the diagonal matrices P and Q must have the same spectrum, and by permuting the $|a_i\rangle\otimes|\psi_i\rangle$ we may assume that $P=Q$. So, we have $P=XPX^\dagger$, i.e., $PX=XP$. Suppose that one of the eigenvalues p_i of P, say p_1, is simple. Then X breaks into a direct sum of two matrices, the first one being just 1×1 matrix. This implies that $|a_1,\psi_1\rangle\propto|b_1,\varphi_1\rangle$. Therefore $|\psi_1\rangle$ must be a product vector, which contradicts part (iii). Hence, each p_i must have multiplicity at least two. Suppose now that P is not a scalar matrix, say $p_1=p_2\neq p_3=p_4$. Then X breaks into a direct sum $X=X_1\oplus X_2$ of two 2×2 matrices. This implies that the states $\Psi=\sum_{i=1}^{2}|a_i\rangle\langle a_i|_{A_1}\otimes|\psi_i\rangle\langle\psi_i|_{A_2A_3}$ and $\Phi=\sum_{i=1}^{2}|b_i\rangle\langle b_i|_{A_2}\otimes|\varphi_i\rangle\langle\varphi_i|_{A_1A_3}$ are equal. So the state $\langle a_1^\perp|\Psi|a_1^\perp\rangle\propto|\psi_2\rangle\langle\psi_2|$ is separable, which is a contradiction with (iii). We conclude that P must be a scalar matrix, and so $\sigma\propto\rho$. This concludes the proof. $\square$

Example 27 To illustrate Proposition 26, we consider the three-qubit PPTES ρ constructed from the well known [17, Eq. (22)] three-qubit UPB

$$|000\rangle,\ |+,1,-\rangle,\ |1,-,+\rangle,\ |-,+,1\rangle, \tag{23}$$

where $|\pm\rangle=(|0\rangle\pm|1\rangle)/2$. The state ρ is the normalized projector whose kernel is spanned by the four product vectors of this UPB. Explicitly, we have

$$\rho\ =\ |+,\psi_1\rangle\langle+,\psi_1|+|-,\psi_2\rangle\langle-,\psi_2|+|0,\psi_3\rangle\langle0,\psi_3|+|1,\psi_4\rangle\langle1,\psi_4|. \tag{24}$$

where

$$|\psi_1\rangle\ =\ \frac{1}{\sqrt{6}}(2|01\rangle+|10\rangle+|11\rangle), \tag{25}$$

$$|\psi_2\rangle\ =\ \frac{1}{\sqrt{6}}(-|01\rangle-2|10\rangle+|11\rangle), \tag{26}$$

$$|\psi_3\rangle\ =\ \frac{1}{\sqrt{3}}(-|01\rangle+|10\rangle+|11\rangle), \tag{27}$$

$$|\psi_4\rangle\ =\ \frac{1}{\sqrt{12}}(3|00\rangle-|01\rangle+|10\rangle+|11\rangle). \tag{28}$$

Note that Eq. (24) is exactly the decomposition Eq. (19) in part (iii) of Proposition 26. Other assertions of that proposition can be easily verified.

We recall from [10] that the two-qutrit PPT state of rank four is entangled if and only if its range is a CES. The analogous result is valid for three-qubit states, see Proposition 26 (ii). These observations and Theorem 15, 22 imply the following important result.

Theorem 28 *Let ρ be a multipartite PPT state of rank $r\leq 4$. If $r<4$ then ρ is separable. If $r=4$ then ρ is entangled if and only if $\mathcal{R}(\rho)$ is a CES.*

In the next section we show that the CES condition can be easily checked. This is based on the so-called Chow form of Segre varieties. It has been shown in [4, Theorem 2] that any normalized three-qubit PPTES ρ such that $r(\rho)=r(\rho^{\Gamma_1})=r(\rho^{\Gamma_2})=4$ is an extreme point of the set of all PPT states. We conclude this section with the following observation.

Proposition 29 *Any three-qubit PPTES of rank four is strongly extreme.*

Proof. Let ρ be a PPTES of rank four. By Theorem 22, ρ is supported on a $3 \otimes 3$ or $2 \otimes 2 \otimes 2$ subsystem. By Theorem 3 (iv), the assertion holds in the former case. For the latter case, let us consider the a PPT state σ such that $\mathcal{R}(\sigma) \subseteq \mathcal{R}(\rho)$. By Proposition 26 (ii), σ must be entangled. Lemma 11 and Theorem 15 imply that $r(\sigma) = 4$. Then Proposition 26 (v) implies that $\rho \propto \sigma$. This completes the proof. $\square$

By Definition 9, any three-qubit PPTES of rank four is also extreme.

Note that neither of Theorem 28 and Proposition 29 holds for PPTES of rank five. For example, let ρ be a two-qutrit PPTES of rank four. Then $\sigma = \rho + e|00\rangle\langle 00|$ is a two-qutrit PPTES of rank five when $e > 0$ is sufficiently small. Evidently $\mathcal{R}(\sigma)$ is not a CES. Furthermore, σ is not extreme and thus not strongly extreme by Definition 9.

VI. CHOW FORMS OF SOME SEGRE VARIETIES

In this section we show how one can test whether the CES condition of Theorem 28 is satisfied. Let us first recall some facts from Algebraic Geometry that we need.

Let $X \subseteq \mathcal{P}$ be an irreducible projective variety embedded in a complex projective space $\mathcal{P}$. If $L \subseteq \mathcal{P}$ is a linear subspace of dimension $\operatorname{Dim} \mathcal{P} - \operatorname{Dim} X$ then it is well known that $L \cap X \neq \emptyset$. On the other hand, the set of all linear subspaces L of dimension $\operatorname{Dim} \mathcal{P} - \operatorname{Dim} X - 1$ which meet X is a closed irreducible hypersurface in the corresponding Grassmannian. This hypersurface is known as the *associated hypersurface* of X. It is defined by an equation $F = 0$, where F is an irreducible homogeneous polynomial in the Plücker coordinates of L known as the *Chow form* of X. It is known that the degree of the polynomial F is eaqual to the degree of the variety X. For more details about the associated hypersurface, see [25, Chapter 3].

If V is a subspace of $\mathcal{H}$ of dimension $d - \sum_i (d_i - 1)$ then V must contain a product vector. This is not true for subspaces of smaller dimension. The case when

$$\operatorname{Dim} V = \delta_1 := d - 1 - \sum_{i=1}^{n} (d_i - 1) \tag{29}$$

is of interest because the set of all vector subspaces of this dimension which contain a product vector form an irreducible hypersurface in the corresponding Grassmannian. This is the associated hypersurface of the Segre variety $\Sigma = \mathcal{P}^{d_1 - 1} \times \cdots \times \mathcal{P}^{d_n - 1} \subseteq \mathcal{P}^{d-1}$ canonically embedded in the projective space $\mathcal{P}^{d-1}$ of $\mathcal{H}$. It is defined by a single homogeneous polynomial equation $F(p_V) = 0$ in the Plücker coordinates

$$p_V = \{p_{i_1 i_2 \ldots i_{\delta_1}} : 0 < i_1 < i_2 < \cdots < i_{\delta_1} \leq d\} \tag{30}$$

of V. The degree of F is equal to the degree of Σ.

Let us give a few examples of Chow forms F of Segre varieties Σ for bipartite quantum systems $M \otimes N$. In all cases F will be given as a determinant of a square matrix of order $\delta = \binom{M+N-2}{M-1}$. The Plücker coordinates of V have to be computed in a product basis, say $|a_i, b_j\rangle$ $(i = 1, \ldots, M;\ j = 1, \ldots, N)$. (See the Example 32.) It is not required that this basis be orthogonal. The basis has to be ordered, and we shall use always the increasing lex ordering: $|a_1, b_1\rangle, |a_1, b_2\rangle, \ldots, |a_1, b_N\rangle, |a_2, b_1\rangle, \ldots, |a_M, b_N\rangle$. We start with the cases where $N = 2$.

For $(M, N) = (2, 2)$ we have

$$F = \begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix}. \tag{31}$$

For $(M, N) = (3, 2)$ we have

$$F = \begin{vmatrix} p_{13} & p_{14} + p_{23} & p_{24} \\ p_{15} & p_{16} + p_{25} & p_{26} \\ p_{35} & p_{36} + p_{45} & p_{46} \end{vmatrix}. \tag{32}$$

For $(M, N) = (4, 2)$ we have

$$F = \begin{vmatrix} p_{135} & p_{136} + p_{145} + p_{235} & p_{146} + p_{236} + p_{245} & p_{246} \\ p_{137} & p_{138} + p_{147} + p_{237} & p_{148} + p_{238} + p_{247} & p_{248} \\ p_{157} & p_{158} + p_{167} + p_{257} & p_{168} + p_{258} + p_{267} & p_{268} \\ p_{357} & p_{358} + p_{367} + p_{457} & p_{368} + p_{458} + p_{467} & p_{468} \end{vmatrix}. \tag{33}$$

This can be generalized to any pair $(M, 2)$.

Proposition 30 *The Chow form F of the Segre variety $\mathcal{P}^{M-1} \times \mathcal{P}^1$ embedded canonically in the projective space $\mathcal{P}^{2M-1}$ is the determinant of the $M \times M$ matrix $B = [b_{ij}]$, whose entry b_{ij} is the sum*

$$b_{ij} = \sum_{P_{ij}} p_{k_1,k_2,\ldots,k_{M-1}} \tag{34}$$

of Plücker coordinates $p_{k_1,k_2,\ldots,k_{M-1}}$ taken over the set P_{ij} of all $\binom{M-1}{j-1}$ sequences $(k_1, k_2, \ldots, k_{M-1})$ obtained from the sequence $(1, 3, \ldots, \widehat{2(M-i)+1}, \ldots, 2M-1)$ by incrementing by one exactly $j-1$ of its terms. (The circumflex indicates that the term $2(M-i)+1$ should be omitted.)

The proof of this proposition is given in Appendix I.

Let us point out that if one has a formula for the Chow form $F = F_{M,N}$ of the Segre variety $\Sigma_{M,N}$ embedded in the projective space $\mathcal{P}_{A,B}$, one can easily find the Chow form $F' = F_{N,M}$ of the Segre variety $\Sigma_{N,M}$ embedded in the projective space $\mathcal{P}_{B,A}$. We shall illustrate this procedure in the case $(M,N) = (3,2)$. The form $F = F_{3,2}$, given by Eq. (32), is a cubic polynomial in the Plücker coordinates p_{ij} computed in the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle, |20\rangle, |21\rangle$ of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Note that this basis consists of bases of the subspaces $|0\rangle \otimes \mathcal{H}_B$, $|1\rangle \otimes \mathcal{H}_B$, $|2\rangle \otimes \mathcal{H}_B$ written in that order. To compute the form F', we shall use the ordered basis $|00\rangle, |10\rangle, |20\rangle, |01\rangle, |11\rangle, |21\rangle$, where the first [last] three vectors form a basis of the subspace $\mathcal{H}_A \otimes |0\rangle$ $[\mathcal{H}_A \otimes |1\rangle]$. Note that the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix} \tag{35}$$

transforms the first basis into the second. The Chow form $F_{2,3}$ is obtained from $F_{3,2}$ by replacing each Plücker coordinate p_{ij} with $p_{\pi(i)\pi(j)}$. Thus for $(M,N) = (2,3)$ we have

$$F_{2,3} = \begin{vmatrix} p_{12} & p_{15} - p_{24} & p_{45} \\ p_{13} & p_{16} - p_{34} & p_{46} \\ p_{23} & p_{26} - p_{35} & p_{56} \end{vmatrix}, \tag{36}$$

where we used the fact that $p_{ji} = -p_{ij}$. Let us compare this result with the one given in [9, Example 23]. By using the quadratic Plücker relations $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$, $p_{14}p_{56} - p_{15}p_{46} + p_{16}p_{45} = 0$ and $p_{24}p_{56} - p_{25}p_{46} + p_{26}p_{45} = 0$, one can easily verify that the polynomial on the left hand side of the equation on top of p. 16 in [9] is equal to $-F_{2,3}$. Since the Chow form is determined only up to a scalar factor, the two results agree.

The most important case for us is $(M,N) = (3,3)$. In that case we have

$$F = \begin{vmatrix} p_{1245} & p_{1346} & p_{2356} & p_{1246} + p_{1345} & p_{1256} + p_{2345} & p_{1356} + p_{2346} \\ p_{1278} & p_{1379} & p_{2389} & p_{1279} + p_{1378} & p_{1289} + p_{2378} & p_{1389} + p_{2379} \\ p_{4578} & p_{4679} & p_{5689} & p_{4579} + p_{4678} & p_{4589} + p_{5678} & p_{4689} + p_{5679} \\ p_{1248} - p_{1257} & p_{1349} - p_{1367} & p_{2359} - p_{2368} & \begin{matrix} p_{1249} - p_{1267} \\ +p_{1348} - p_{1357} \end{matrix} & \begin{matrix} p_{1259} - p_{1268} \\ +p_{2348} - p_{2357} \end{matrix} & \begin{matrix} p_{1359} - p_{1368} \\ +p_{2349} - p_{2367} \end{matrix} \\ p_{1458} - p_{2457} & p_{1469} - p_{3467} & p_{2569} - p_{3568} & \begin{matrix} p_{1459} - p_{2467} \\ +p_{1468} - p_{3457} \end{matrix} & \begin{matrix} p_{1568} - p_{2567} \\ +p_{2459} - p_{3458} \end{matrix} & \begin{matrix} p_{1569} - p_{3468} \\ +p_{2469} - p_{3567} \end{matrix} \\ p_{1578} - p_{2478} & p_{1679} - p_{3479} & p_{2689} - p_{3589} & \begin{matrix} p_{1579} - p_{2479} \\ +p_{1678} - p_{3478} \end{matrix} & \begin{matrix} p_{1589} - p_{2489} \\ +p_{2678} - p_{3578} \end{matrix} & \begin{matrix} p_{1689} - p_{3489} \\ +p_{2679} - p_{3579} \end{matrix} \end{vmatrix}. \tag{37}$$

The Chow form of the Segre variety of the system $2 \otimes 2 \otimes 2$ of three qubits is given explicitly in [48, Proposition 4.10]. Some misprints in the determinant above that proposition have been corrected in [20, Example 22, Eq. (23)]. We point out that the determinantal formula in these references is written in terms of the dual Plücker coordinates. When translated into ordinary Plücker coordinates, the Chow form is given by the following determinant:

$$F = \begin{vmatrix} p_{1235} & p_{1237} & p_{1567} & p_{3567} & p_{1257} - p_{1356} & p_{1367} - p_{2357} \\ p_{1246} & p_{1248} & p_{2568} & p_{4568} & p_{1268} - p_{2456} & p_{1468} - p_{2458} \\ p_{1345} & p_{1347} & p_{1578} & p_{3578} & p_{1358} - p_{1457} & p_{1378} - p_{3457} \\ p_{2346} & p_{2348} & p_{2678} & p_{4678} & p_{2368} - p_{2467} & p_{2478} - p_{3468} \\ p_{1236} + p_{1245} & p_{1238} + p_{1247} & p_{1568} + p_{2567} & p_{3568} + p_{4567} & \begin{matrix} p_{1258} - p_{1456} \\ +p_{1267} - p_{2356} \end{matrix} & \begin{matrix} p_{1368} - p_{2358} \\ +p_{1467} - p_{2457} \end{matrix} \\ p_{1346} + p_{2345} & p_{1348} + p_{2347} & p_{1678} + p_{2578} & p_{3678} + p_{4578} & \begin{matrix} p_{1368} - p_{1467} \\ +p_{2358} - p_{2457} \end{matrix} & \begin{matrix} p_{1478} - p_{3458} \\ +p_{2378} - p_{3467} \end{matrix} \end{vmatrix}. \tag{38}$$

The problem of deciding whether a bipartite PPT state ρ of rank four is separable has been solved in [9, Theorem 22]. The answer is affirmative if and only if $\mathcal{R}(\rho)$ contains at least one product vector. The proof easily reduces to the case when ρ is a 3×3 state. In that case we can improve the mentioned theorem. By Theorem 28, the analogous result is valid for the PPT three-qubit states of rank four. Thus we have the following result.

Theorem 31 *A 3×3 or $2\times 2\times 2$ state ρ of rank four is separable if and only if ρ is PPT and the Plücker coordinates of $\mathcal{R}(\rho)$ satisfy the equation $F=0$ where F is the Chow form (37) or (38), respectively.*

We illustrate this theorem by the following example.

Example 32 ($M=N=3$) Let $\rho=\sum_i |\psi_i\rangle\langle\psi_i|$ where $|\psi_1\rangle = |00\rangle + a|11\rangle$, $|\psi_2\rangle = a|01\rangle + |10\rangle + b|21\rangle$, $|\psi_3\rangle = |11\rangle + b|20\rangle + |22\rangle$, $|\psi_4\rangle = |12\rangle + |21\rangle$ and a,b are complex parameters. It is easy to verify that this is always a PPT state of rank four. The range of ρ is spanned by the rows of the matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & b & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{39}$$

It is now easy to compute the Plücker coordinates p_{ijkl} of $\mathcal{R}(\rho)$. Recall that p_{ijkl} is the determinant of the full submatrix of R in columns i,j,k,l. We find that

$$\begin{aligned}
&p_{1456} = p_{1458} = 1,\\
&p_{1469} = p_{1489} = -1,\\
&p_{1256} = p_{1258} = p_{4569} = p_{4589} = a,\\
&p_{1269} = p_{1289} = -a,\\
&p_{1478} = p_{1568} = p_{1689} = b,\\
&p_{1467} = -b,\\
&p_{2569} = p_{2589} = a^2,\\
&p_{1278} = p_{4567} = p_{5689} = ab,\\
&p_{1267} = p_{4578} = -ab,\\
&p_{1678} = -b^2,\\
&p_{2567} = a^2 b,\\
&p_{2578} = -a^2 b,\\
&p_{5678} = -ab^2,
\end{aligned}$$

and all other are 0. By plugging in these values into Eq. (37), we obtain that $F=-a^4b^4$. Hence, ρ is separable if and only if $ab=0$. $\square$

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[1] A. Acin, M.L. Almeida, R. Augusiak, and N. Brunner, Guess your neighbour's input: no quantum advantage but an advantage for quantum theory, quant-ph/1205.3076 (2012).

[2] A. Acin, D. Bruss, M. Lewenstein, and A. Sanpera, Classification of mixed three-qubit states, Phys. Rev. Lett. **87**, 040401 (2001).

[3] E. Alfsen and F. Shultz, Unique decompositions, faces, and automorphisms of separable states, J. Math. Phys. **51**, 052201 (2010).

[4] R. Augusiak, J. Grabowski, M. Kus, and M. Lewenstein, Searching for extremal PPT entangled states, Optics Commun. **283** (2010), 805-813.

[5] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters, Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).

[6] C.H. Bennett, D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, and B.M. Terhal, Unextendible product bases and bound entanglement, Phys. Rev. Lett. **82** (1999), 5385-5388.

[7] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).

[8] Lin Chen, E. Chitambar, K. Modi, G. Vacanti, Detecting multipartite classical states and their resemblances, Phys. Rev. A **83**, 020101(R) (2011).

[9] Lin Chen and D.Ž. Đoković, Distillability and PPT entanglement of low rank quantum states, J. Phys. A: Math. Theor. **44**, 285303 (2011), (26pp).

[10] Lin Chen and D.Ž. Đoković, Description of rank four entangled states of two qutrits having positive partial transpose, J. Math. Phys. **52**, 122203 (2011), (27pp).

[11] Lin Chen and D.Ž. Đoković, Equivalence classes and canonical forms for two-qutrit entangled states of rank four having positive partial transpose, J. Math. Phys. **53**, 102205 (2012), (14pp).

[12] Lin Chen and D.Ž. Đoković, Qubit-qudit states with positive partial transpose, Phys. Rev. A **86**, 062332 (2012).

[13] Lin Chen and D.Ž. Đoković, Properties and construction of extreme bipartite states having positive partial transpose, quant-ph/1203.1364 (2012).

[14] Lin Chen and D.Ž. Đoković, Dimensions, lengths and separability in finite-dimensional quantum systems. Preprint, quant-ph/1206.3775v3 (2012).

[15] H.-S. Choi and S.-H. Kye, Facial structures for separable states, J. Korean Math. Soc. **49**, 623 (2012).

[16] M. Christandl and A. Winter, "Squashed entanglement": An additive entanglement measure, J. Math. Phys.**45**, 829 (2004).

[17] D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, and B.M. Terhal, Unextendible product bases, uncompletable product bases and bound entanglement, Commun. Math. Phys. **238** (2003), 379-410.

[18] D.P. DiVincenzo, B.M. Terhal, and A.V. Thapliyal, Optimal decomposition of barely separable states, J. Mod. Opt. **47** (2000), 377-385.

[19] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Complete family of separability criteria, Phys. Rev. A **69**, 022308 (2004).

[20] M. Drton, B. Sturmfels, and S. Sullivant, Algebraic factor analysis: tetrads, pentads and beyond, arXiv:math/0509390v2 [math.ST] 3 Nov 2006.

[21] R. Duan, Y. Feng, Y. Xin, and M. Ying, Distinguishability of quantum states by separable operations, IEEE. Trans. Inf. Theory**55**, 1320 (2009).

[22] W. Dür, G. Vidal, and J.I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).

[23] S.-M. Fei, X.-H. Gao, X.-H. Wang, Z.-X. Wang, and K. Wu, Separability and entanglement in $\mathbf{C}^2 \otimes \mathbf{C}^3 \otimes \mathbf{C}^N$ composite quantum systems, Phys. Rev. A **68**, 022315 (2003).

[24] O.S. Garberg, B. Irgens, and J. Myrheim, Extremal states of positive partial transpose in a system of three qubits, quant-ph/1212.5036 (2012).

[25] I. Gelfand, M. Kapranov, and A. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[26] L. Gurvits, Classical deterministic complexity of Edmonds' problem and quantum entanglement, in Proceedings of the Thirty-fifth annual ACM Symposium on Theory of Computing, San Diego, California, June 9-11, 2003, ACM Press, New York, p. 10 (2003).

[27] K.-C. Ha, and S.-H. Kye, Construction of entangled states with positive partial transposes based on indecomposable positive linear maps. Phys. Lett. A, **325**, 315-323 (2004).

[28] K-C. Ha and S-H. Kye, Separable states with unique decompositions, preprint. arXiv:1210.1088v2.

[29] K-C. Ha and S-H. Kye, Exposedness of Choi type entanglement witnesses and applications to lengths of separable states, preprint. arXiv:1211.5675v1.

[30] J. Harris, Algebraic Geometry, A First Course, Springer-Verlag, New York (1992).

[31] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A **223** (1996), 1-8.

[32] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transpose, Phys. Lett. A **232** (1997), 333-339.

[33] P. Horodecki, M. Lewenstein, G. Vidal, and I. Cirac, Operational criterion and constructive checks for separability of low rank density matrices, Phys. Rev. A **62**, 032310 (2000).

[34] P. Horodecki, J.A. Smolin, B.M. Terhal, A.V. Thapliyal, Rank two bipartite bound entangled states do not exist, Theoretical Computer Science **292** (2003), 589-596.

[35] S. Karnas and M. Lewenstein, Separability and entanglement in $C^2 \otimes C^2 \otimes C^N$ composite quantum systems, Phys. Rev. A **64**, 042313 (2001).

[36] Y. H. Kiem, S.-H. Kye, and J. Lee, Existence of product vectors and their partial conjugates in a pair of spaces, J. Math. Phys. **52**, 122201 (2012).

[37] W.C. Kim and S.-H. Kye, Extreme rays in $3 \otimes 3$ entangled edge states with positive partial transposes, Phys. Lett. A **369**, 16 (2007).

[38] B. Kraus, J.I. Cirac, S. Karnas, and M. Lewenstein, Separability in $2 \times N$ composite quantum systems, Phys. Rev. A **61**,

062302 (2000).
[39] S.-H. Kye and H. Osaka, Classification of bi-qutrit PPT entangled states by their ranks, quant-ph/1202.1699 (2012).
[40] J.M. Leinaas, J. Myrheim and E. Ovrum, Extreme points of the set of density matrices with positive partial transpose, Phys. Rev. A **76**, 034304 (2007).
[41] R. Lockhart, Optimal ensemble length of mixed separable states, J. Math. Phys. **41**, (2000), 6766-6771.
[42] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[43] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77** (1996), 1413-1415.
[44] M. Piani, P. Horodecki, and R. Horodecki, No-local-broadcasting theorem for multipartite quantum correlations, Phys. Rev. Lett. **100**, 090502 (2008).
[45] A. Sanpera, R. Tarrach, and G. Vidal, Local description of quantum inseparability, Phys. Rev. A **58**, 826 (1998).
[46] M. Seevinck and G. Svetlichny, Bell-type inequalities for partial separability in N-particle systems and quantum mechanical violations, Phys. Rev. Lett. **89**, 060401 (2002).
[47] L. Skowronek, Three-by-three bound entanglement with general unextendible product bases, J. Math. Phys. **52**, 122202 (2011), (32pp).
[48] B. Sturmfels, Solving Systems of Polynomial Equations, Regional Conference Series in Mathematics No. 97, 2002.
[49] G. Toth and O. Guhne, Detecting genuine multipartite entanglement with two local measurements, Phys. Rev. Lett. **94**, 060501 (2005).
[50] R.F. Werner, Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).
[51] W.K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

Appendix I: Proof of Proposition 30

We shall now derive Proposition 30 from Theorem 3.19 and Proposition 3.21 of [25, Chapter 14].

Let us first recall these results and introduce the necessary notation. For convenience we set $[n] = \{1, 2, \dots, n\}$ for any positive integer n. We consider the bipartite system $M \otimes N$. We are interested in subspaces V of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $(M-1)(N-1)$. Such V is defined by a system of $M+N-1$ independent linear homogeneous equations $\sum_{i,j,k} a_{ijk}\xi_{ik} = 0$, $j \in [M+N-1]$. We assume here that the vectors $|x\rangle \in V$ are given as linear combinations of standard basis vectors $|x\rangle = \sum_{i,k} \xi_{ik}|i-1\rangle \otimes |k-1\rangle$. The coefficients a_{ijk} form a 3-dimensional matrix $A = [a_{ijk}]$, where $i \in [M]$, $j \in [M+N-1]$ and $k \in [N]$. We can rearrange the entries of A to obtain an ordinary $(M+N-1)\times MN$ matrix A^{13}. First, let us note that any $r \in [MN]$ can be uniquely written as $r = (i_r - 1)N + k_r$ with $i_r \in [M]$ and $k_r \in [N]$. We define $\tilde{r} = (i_r, k_r)$ for $r \in [MN]$. Then we define the (j, r)th entry of A^{13} to be a_{i_r,j,k_r} where $\tilde{r} = (i_r, k_r)$.

For any sequence $r_1, r_2, \dots, r_{M+N-1}$ in $[MN]$, we denote by $q_{r_1,r_2,\dots,r_{M+N-1}}$ the determinant of the square matrix formed from the columns of A^{13} writen in the indicated order: first column r_1, then column r_2, etc. The *dual Plücker coordinates* of V are the determinants $q_{r_1,r_2,\dots,r_{M+N-1}}$ with $r_1 < r_2 < \dots < r_{M+N-1}$. There is a simple relation between the dual and ordinary Plücker coordinates of V, see Eq. (1.6) in [25, Chapter 3]. In our case it has the following form

$$q_{r_1,r_2,\dots,r_{M+N-1}} = \varepsilon p_{r'_1,r'_2,\dots,r'_{(M-1)(N-1)}}, \tag{40}$$

where the indexes $r'_1, \dots, r'_{(M-1)(N-1)}$ are arranged in increasing order and form the complement of $\{r_1, \dots, r_{M+N-1}\}$ in $[MN]$. The symbol $\varepsilon = \pm 1$ is the sign of the permutation $r'_1, \dots, r'_{(M-1)(N-1)}, r_1, \dots, r_{M+N-1}$.

Another piece of notation that we need is

$$\Delta^{p-1}(m) = \{\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{Z}_+^p : \alpha_1 + \dots + \alpha_p = m\}, \tag{41}$$

where $\mathbf{Z}_+$ is the set of nonnegative integers.

As shown in Theorem 3.19 of [25, Chapter 14], the Chow form F of the Segre variety $\mathcal{P}^{M-1} \times \mathcal{P}^{N-1}$ in $\mathcal{P}^{MN-1}$ is given by $F = \det B$, where $B = [b_{\alpha,\beta}]$ is a square matrix of order δ, and $\alpha \in \Delta^{M-1}(N-1)$ and $\beta \in \Delta^{N-1}(M-1)$. Furthermore, there is also a formula for the matrix entries $b_{\alpha,\beta}$. To state this formula, we need to introduce the complete bipartite graph whose vertex set is the disjoint union of $[M]$ and $[N]$ and the edge set is the Cartesian product $[M] \times [N]$. Any spanning tree, Ω, of this graph consists of $M + N - 1$ edges. Moreover, Ω can be written uniquely as

$$\Omega = \{(1, k_1), \dots, (M, k_M), (i_2, 2), \dots, (i_N, N)\}. \tag{42}$$

We remark that the indexes k and i are not arbitrary. For instance, since $|\Omega| = M + N - 1$ we must have $k_{i_r} \neq r$ for $r \in \{2, 3, \dots, N\}$. Then we define $[\Omega] = q_{r_1,r_2,\dots,r_{M+N-1}}$, where

$$r_s = (s-1)N + k_s, \quad s \in [M]; \quad r_{M+s} = (i_s - 1)N + s, \quad s = 2, 3, \dots, N. \tag{43}$$

For $(\alpha,\beta) \in \Delta^{M-1}(N-1) \times \Delta^{N-1}(M-1)$ with $\alpha = (\alpha_1,\dots,\alpha_M)$ and $\beta = (\beta_1,\dots,\beta_N)$, we shall write $\Omega \vdash (\alpha,\beta)$ if $\alpha_r = |\{s : i_s = r\}|$ for $r \in [M]$ and $\beta_s = |\{i : k_i = s\}|$ for $s \in \{2,3,\dots,N\}$. With this notation, Proposition 3.21 of [25, Chapter 14] asserts that the entries $b_{\alpha,\beta}$ are given by the formula

$$b_{\alpha,\beta} = \sum_{\Omega \vdash (\alpha,\beta)} [\Omega]. \tag{44}$$

We can now prove Proposition 30. By hypothesis we have $N = 2$. Thus $\Delta^{M-1}(N-1) = \{\varepsilon_i : i \in [M]\}$ where the ith entry of ε_i is 1 and all other 0. Note that $\Delta^{N-1}(M-1) = \{(M-1-s,s) : s = 0,1,\dots,M-1\}$. Let us compute the entry $b_{\alpha,\beta}$ for $\alpha = \varepsilon_u$ and $\beta = (M-1-v,v)$. Let $\Omega := \{(1,k_1),\dots,(M,k_M),(i_2,2)\} \vdash (\alpha,\beta)$. By using the definition of the relation "$\vdash$" and the fact that $\alpha = \varepsilon_u$, we infer that $i_2 = u$. As mentioned above, we must have $k_{i_2} \neq 2$, and so $k_u = 1$. Moreover, the set $S = \{i : k_i = 2\}$ has cardinality v. Clearly, for fixed u and v, Ω is determined uniquely by the set S. Note that S is subject only to the conditions that $|S| = v$ and $u \notin S$. Consequently, there are exactly $\binom{M-1}{v-1}$ maximal trees Ω such that $\Omega \vdash (\alpha,\beta)$. For convenience, we shall write $[r_1,\dots,r_{M+1}] = q_{r_1,\dots,r_{M+1}}$, and we shall use similar notation for ordinary Plücker coordinates. Next note that

$$\begin{aligned} [\Omega] &= [k_1, 2+k_2, 4+k_3, \dots, 2M-2+k_M, 2u] \\ &= (-1)^{M-u}[k_1, 2+k_2, \dots, 2u-4+k_{u-1}, 2u-1, 2u, 2u+k_{u+1}, \dots, 2M-2+k_M]. \end{aligned} \tag{45}$$

Passing to the ordinary Plücker coordinates, we obtain that

$$[\Omega] = (-1)^{M-u}\varepsilon[3-k_1, 5-k_2, 7-k_3, \dots, 2u-1-k_{u-1}, 2u+3-k_{u+1}, \dots, 2M+1-k_M], \tag{46}$$

where ε is the sign of the permutation

$$\begin{aligned} \sigma \;=\; & 3-k_1, 5-k_2, 7-k_3, \dots, 2u-1-k_{u-1}, 2u+3-k_{u+1}, \dots, 2M+1-k_M, \\ & k_1, 2+k_2, 4+k_3, \dots, 2u-4+k_{u-1}, 2u-1, 2u, 2u+k_{u+1}, \dots, 2M-2+k_M. \end{aligned} \tag{47}$$

Since F is defined only up to a scalar factor, we can change the sign of any row or column of B if necessary. Hence we can ignore the factor $(-1)^{M-u}$ in Eq. (45) and replace σ by the permutation

$$\begin{aligned} \sigma' \;=\; & k_1, 3-k_1, 2+k_2, 5-k_2, 4+k_3, 7-k_3, \dots, 2u-4+k_{u-1}, 2u-1-k_{u-1}, 2u-1, 2u, \\ & 2u+k_{u+1}, 2u+3-k_{u+1}, \dots, 2M-2+k_M, 2M+1-k_M. \end{aligned} \tag{48}$$

Since the sign of σ' is $(-1)^v$, we may assume that

$$[\Omega] = p_{3-k_1, 5-k_2, 7-k_3, \dots, 2u-1-k_{u-1}, 2u+3-k_{u+1}, \dots, 2M+1-k_M}. \tag{49}$$

We conclude that after suitable permutation of rows and columns, and multiplying some rows and columns with -1, the matrix B used in this proof becomes equal to the transpose of the matrix B defined in the proposition. This completes the proof.